VIBRATION OF VISCOUSLY DAMPED LINEAR DYNAMIC SYSTEMS

Thesis by

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ABSTRACT

A general theory of vibration of damped linear dynamic systems is given. The limitations on the use of the usual normal mode theory in determining the response of damped systems were first studied systematically by Caughey when he derived necessary and sufficient conditions for the uncoupling of systems in N-space. Systems which cannot be uncoupled in N-space may still be solvable by modal methods on transforming them to 2N-space and using the results of Foss. However there exist systems which cannot be solved by the usual modal techniques in either N-space or 2N-space. Such systems which include some passive physically realizable systems require the general theory for a complete determination of their motion. For weakly coupled systems the simple perturbation analysis presented gives surprisingly accurate approximations to the actual response of the systems. In any design problem questions of stability arise, particularly when dealing with non symmetric systems, and therefore a discussion on the stability of these systems is given.

The second part of the thesis is concerned with linear continuous systems. Exactly solvable continuous systems are rare and in general recourse must be had to numerical methods. The interchangeability of the differential and integral formulation of continuous systems is noted. As in the discrete systems constructive necessary and sufficient conditions are derived for a damped system to possess the same set of complete eigenfunctions as the undamped system. In

the discretization of continuous systems the main problem of practical interest is the error bounds on the solution of these discrete approximations when compared to the exact solution. Unfortunately the literature is very poor in this area but what is known is applied to the systems under discussion.

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INTRODUCTION

Modern mechanics came into being shortly after the Renaissance and to this day forms one of the cornerstones in the disciplines of engineering and the physical sciences. In contrast to the earlier development of mechanics at the time of Archimedes, the rapid growth of the science during the seventeenth, eighteenth, and nineteenth centuries is indeed impressive. This growth must be attributed in large part to the constant interactions of mathematics and mechanics during these centuries. There was no imbalance in this interaction for one can truly say that without the stimulus of mechanics the development of mathematics would have been considerably slower.

It is in modern mechanics that one finds the foundations for the study of the vibrations of linear systems. Newton,[1] in his Principia, first formulated the equations of motion of a mass in a force field. Lagrange[2] developed the general theory of the small oscillation of conservative vibrating systems in his book, Mécanique Analytique. As early as 1753 Daniel Bernoulli had introduced one of the fundamental concepts in the analysis of linear systems, namely - that of superposition. Routh[3] by 1877 had succeeded in obtaining the solution of the small oscillation of conservative systems by means of normal mode techniques. Finally Lord Rayleigh[4] tackled the problem of damped systems and used the concept of the dissipation function with great success.

Rayleigh worked with both discrete and continuous systems and seems to have been the first to approximate continuous systems by discrete systems. Due to the comparative ease of conceptualization and the more readily available mathematical tools, considerably more work has been done on the discrete system than on the continuous system. To trace the historical development of the analysis of linear systems in the last one-hundred years, the work done on discrete systems will first be treated. This will be followed by a state-of-the-art discussion of continuous systems. Against this background of the historical development, the contribution of this work to the theory of linear vibrating systems will be noted. At this point, it is important to realize that the thesis is concerned with time invariant linear systems. A linear system is one in which, if y_1 and y_2 are the responses to the excitations x_1 and x_2 respectively, then $a_1y_1 + a_2y_2$ is the response to the excitation $a_1x_1 + a_2x_2$, where a_1 and a_2 are arbitrary constants. A time invariant linear system is simply a linear system the elements of which have responses that are invariant under translation in time, i.e., if $x(t)$ is the output of an element from the input $y(t)$, then, if this element is time invariant, $x(t + \tau)$ is the output to the input $y(t + \tau)$.

Around Rayleigh's time, mathematicians became interested in matrix analysis and it was soon obvious that matrix notation was ideally suited for use in the analysis of lumped parameter time invariant systems. Many of the complexities in Rayleigh's work were eliminated once the problems were formulated as matrix problems.

At the turn of this century most physicists were satisfied with their understanding of the vibrations of lumped parameter systems. They realized that not all such systems can be solved by modal methods, as understood by Routh, but apparently felt that Rayleigh's intuitive observation about the negligible effect of the small coupling damping terms in the uncoupled equations of motion effectively completed the theory for all interesting physical problems. After World War I, engineers began to analyse the vibrations of many complex structures including, for example, airplane wings, frames and rotors of high speed rotating machinery. It is extremely tedious, using Rayleigh's techniques, to calculate, directly, the normal modes of systems with more than three degrees of freedom. However, iterative methods were developed at this time to rapidly approximate the normal modes of a system. With the advent of large memory digital computers many of these iterative techniques are still used to solve very large order systems.

Despite many advances made, there was one notable gap in the theory of discrete linear damped systems. This was the question of discovering what type of damping matrix would allow the system to be uncoupled in N space. Caughey[5] in 1958 derived necessary and sufficient conditions for the uncoupling of damped systems in N space. A further step forward was made by Foss[6] when he formulated the damped problem in 2N-space and succeeded in deriving orthogonality conditions and solutions of systems not solvable by modal methods in N-space.

It was around this time that the author[7] became interested in the discrete problem. After some work under Dr. Caughey it became apparent that not all systems, even all passive systems, were capable of solution by modal methods. Thus, the justification of the general theory of vibration of damped linear systems as given in Chapter I. The great advantage of the general theory is that it does not depend on having symmetric sign definite matrices as did the earlier theory and so may be readily used in such problems as flutter analysis and systems possessing gyroscopic motions. Needless to say, all such systems can in fact be solved by other methods, e. g. , by integral transform techniques - but with greater computational difficulty. One of the main virtues of the modal method, besides its relative simplicity of computation, is the physical insight it gives into the synthesis of the system under discussion.

In passing, it should be remarked that normal mode methods, as such, do not play as big a role in the analysis of electrical circuits as they do in the analysis of mechanical systems. This is due to the development of specialized techniques of analysis, such as 4 pole parameter methods, very suitable to the standardized sections common in electrical work. In filter analysis and transmission line analysis many of the techniques used depend entirely on symmetries not usually found in mechanical systems. However, the work of the electrical engineer in the area of feedback control has added greatly to the understanding of linear systems. Although the concept of transfer function is valid for mechanical vibrations, it is rarely used.

But in the area of stability much useful work has been done by the practitioners of the servo-mechanism art. Until recently the methods of Routh and Hurwitz[8] were the usual techniques used to determine the stability of linear systems. However, the work of Liapunov[9,10,11] has spurred great interest in the stability problem and within the last few years many theorems concerning sufficiency conditions for the stability of systems have been proved. It is interesting to note that considerably more work has been done and sharper results are available for linear time varying systems than for the linear time invariant systems. In the present work some results of applying Liapunov's Second or Direct Method to linear time invariant systems are given.

In many engineering problems a more or less qualitative type analysis is extremely valuable. This is so because generally speaking an exact analysis of a complex structure is more costly and rarely warranted by the accuracy of the parameters used. In earthquake engineering the responses of buildings to earthquake forces are needed. However, the design engineer cannot determine either from his design or experimentally the precise parameters of his structure. It is for reasons of this type that the perturbation analysis developed in this work is presented. This analysis makes it possible to quickly determine the approximate response of a system. In the same vein, some results on the bounds of eigenvalues are noted.

Returning now to the historical development of continuous systems one is immediately struck by the place of Sturm and Liouville[12] in the development. Essentially, they formulated a

theory for self adjoint differential operators quite analogous to the then existing theory for discrete systems. It was by no means a simple transfer of ideas from the discrete to the continuous, for functional analysis is, by its very nature, considerably more difficult than matrix analysis. Nowadays mathematicians have succeeded in developing a very general theory of linear operators, of which matrices, linear differential operators and linear integral operators are examples, but unfortunately its very generality precludes its use in engineering analysis. Without in any way trying to detract from the work of the professional mathematician, it is well to remember that to the mechanician, mathematics is only a tool, and in the final analysis it is the physical insight into the problem that counts.

By the use of the Sturm-Liouville theory many self adjoint second order differential systems can in fact be solved exactly. Self adjointness roughly corresponds to symmetry and so far not much work has been done on non-self adjoint systems. In the continuous problem one is quickly faced with the question of numerical integration. Although there exist numerous methods of numerical integration, very little research has been reported on the error bounds in the numerical integration of the type of differential equation of interest in this work. This is somewhat upsetting, in the sense that until the numerical analysists are able to solve the problem of stability, convergence and error bounds for general equations the theory of the continuous system will be incomplete by that amount.

In the last fifty years a very beautiful and complete theory of integral equations has emerged. Due to the interchangeability of the integral and differential formulations of the continuous system, greater insight into these systems has been obtained by the use of this theory. Some work is available on the numerical solution of integral equations. This work is reported and applied to the systems under discussion. Again the question of error bounds on the solution arises but, as before, very little work on this aspect of the problem has been done. However, there is an adequate theory available for integral equations with symmetric kernels and this has been used to advantage.

There are, of course, some continuous systems which are exactly solvable. In the past the greatest emphasis was on undamped solvable systems. By extending the ideas of Caughey to the continuous problem, it is possible to develop a theory of classically damped continuous systems. As in the discrete case constructive necessary and sufficient conditions are given to test whether a system is classically damped. Moreover, a sufficient criterion for the form of the damping term is given so that the system is classical. In the engineering literature the usual method of solving continuous systems numerically is to approximate such systems by some physically equivalent discrete system. A critical analysis of this approach, as distinct from the numerical analysist's attack on the problem, is given and some general results for uniform and non-uniform vibrating sections are noted.

The range of problems capable of being solved by the methods

discussed in this work is very wide indeed. By the addition of concepts from stochastic processes, the response of linear systems under random excitation was determined by Caughey.[13] The results are not directly useful in non-linear analysis except where it is possible to use linear approximations to the non-linear system.

CHAPTER 1

DISCRETE SYSTEMS — GENERAL THEORY

Introduction

In this chapter a general review of the well known results in the analysis of lumped parameter time invariant linear systems is given. The necessity for Caughey's classification of systems into classically damped and non-classically damped systems is seen immediately. Constructive necessary and sufficient conditions for systems to be classical are derived. Foss's transformation of the problem to 2N-space is noted. Due to the form of the matrices in 2N-space it is seen that not all problems can be solved by the usual modal methods. The existence of the possibility of matrices whose Jordan canonical form is not strictly diagonal sheds new light on the synthesis of linear damped structures. The question of exciting pure modes is discussed. Finally, a brief section deals with practical computer programs for determining the response of these systems.

It is well to note that although the theory was developed with the solution of the discrete system in mind, it is useful in the numerical solution of continuous systems.

Theory

In the analysis of the vibration of any complex mechanical structure the first problem is to derive the equations of motion. To do this, one has to assign coordinates to the system and by application of suitable physical laws derive relations between the coordinates to

determine the motion. Not all physical systems are so constituted that their equations of motion can be written in symmetric form, e. g., gyroscopic systems. However, the equations describing the behavior of passive physically realizable linear systems can always be transformed to a symmetric set of equations. Moreover, the use of either energy or variational methods in the derivation of the equations of motion of these latter systems results directly in symmetric equations. Using the more direct approach of Newton's Second Law may, even with passive systems, result in a non symmetric set of equations.

Lagrange's Equations,[14] one of the energy methods available, are particularly adapted to the formulation of the equations of motion of complicated dynamic systems. Among the advantages of using Lagrange's Equations, over the direct application of Newton's Laws are:

1) The possibility of selecting a set of coordinates which may considerably simplify the algebraic and numerical work involved in solving for the displacements of the system.
2) The equations of motion are derived in exactly the same way for all sets of coordinates.
3) Only the potential and kinetic energies of the various elements are needed and so there is no difficulty about the algebraic signs of the displacement and velocities.

Before using Lagrange's Equations it is necessary to define:

1) Generalized Coordinates (q_i): A set of independent co-

ordinates used to completely describe the motion of the system.

2) Holonomic System: A system so constrained that its number of degrees of freedom equals the number of coordinates required to completely specify its motion.

3) Non-holonomic System: A system so constrained that the number of degrees of freedom is less than the number of coordinates required to completely specify its motion.

The generalized coordinates are not necessarily confined to be cartesian coordinates and may in fact be angles, quantities with the dimensions of energy or angular momentum, etc. They are usually chosen with regard to the geometry of the system. If the generalized coordinates are not independent or if the system is non-holonomic it is not possible to use Lagrange's Equations in the straight forward manner presented below. However the application of Lagrange's Equations to holonomic systems avoids much of the geometrical considerations usual to the direct use of Newton's Laws.

Derivation of the Equations of Motion of a Holonomic System With N Degrees of Freedom using Lagrange's Equations.

Let q_i, $i=1, 2, \ldots N$ be an independent set of generalized coordinates, used to specify completely the motion of the system. Lagrange's Equation in the usual form is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} T - \frac{\partial}{\partial q_i} T = Q_i \qquad i=1, 2, \ldots N. \tag{1.1}$$

Where T is the kinetic energy of the system of masses, the dot (.)

over the q_i represents differentiation w. r. t. time.

Q_i is the Generalized Force and is defined from the increment of work done by the forces of the system when it moves through displacement dq_i

$$dw = \sum_{j=1}^{N} Q_j \, dq_j \tag{1. 2}$$

From Eq. (1. 2)

$$\frac{\partial w}{\partial q_i} = Q_i \qquad i=1, 2, \ldots N \tag{1. 3}$$

If the masses are in a conservative force field of potential energy V then

$$Q_i = - \frac{\partial V}{\partial q_i} \qquad \text{and} \quad \frac{\partial V}{\partial \dot{q}_i} = 0 \qquad i=1, 2, \ldots N$$

In this case Eq. (1. 1) reduces to

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} L = 0 \tag{1. 4}$$

where L = T-V the Lagrangian function of the system. If the force field consists of a conservative part and a non-conservative part, e. g., in a system of masses coupled by springs (conservative) and viscous dampers (non-conservative), it may be advantageous to write $Q_i = -\partial V / \partial q_i + F_i \quad i=1, 2, \ldots N$ where V represents the potential energy of the conservative field and F_i the contribution of the non-conservative field to Q_i. In this case (1. 1) may be rewritten as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \tag{1. 5}$$

For linear time invariant systems the form of T and V are as follows

$$T = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} m_{p\,ij}\, \dot{q}_i\, \dot{q}_j \tag{1.6}$$

Where m_{pij} are constants with dimensions such that T has the dimensions of energy$\left(ML^2T^{-2}\right)$.

$$V = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} k_{pij}\, q_i\, q_j \tag{1.7}$$

where k_{pij} are constants with dimensions such that V has the dimensions of energy (ML^2T^{-2}). Eqs. (1.6) and (1.7) may be conveniently rewritten in vector matrix notation as follows

$$\begin{aligned} T &= \frac{1}{2} \left\{\dot{q}\right\}^T \left[M\right]_p \left\{\dot{q}\right\} \\ V &= \frac{1}{2} \left\{q\right\}^T \left[K\right]_p \left\{q\right\} \end{aligned} \tag{1.8}$$

where $\left[M\right]_p$ and $\left[K\right]_p$ are N x N matrices with ij^{th} elements m_{pij} and k_{pij} respectively. $\left\{q\right\}$ and $\left\{\dot{q}\right\}$ are N x 1 column vectors with il elements q_i and $\dot{q}_i$ respectively. Superscript T indicates the transpose of the matrix or the vector.

In linear time invariant systems the form of F_i is restricted to

$$\left\{F\right\} = -\left[M\right]_{ss}\left\{\ddot{q}\right\} - \left[C\right]\left\{\dot{q}\right\} - \left[K\right]_{ss}\left\{q\right\} + \left\{f(t)\right\} \tag{1.9}$$

where $\left\{F\right\}$ is a N x 1 column vector with il element F_i, the generalized force at coordinate q_i, i=1, 2, ... N, where $\left[M\right]_{ss}$ is a N x N skew symmetric matrix with ij^{th} element $m_{ss\,ij}$

$$\therefore \quad m_{ss\,ij} = -m_{ss\,ij} \qquad i \neq j$$
$$m_{ss\,ii} = 0 \qquad i, j = 1, 2, \ldots N. \tag{1.10}$$

$\left[K\right]_{ss}$ is a N x N skew symmetric matrix with ij^{th} element $k_{ss\,ij}$

$\left[C\right]$ is a general N x N matrix

$\left\{f(t)\right\}$ is a N x 1 column vector with il element $f_i(t)$, the actual external force applied at coordinate q_i.

The constant elements of the three matrices $\left[M\right]_{ss}$, $\left[C\right]$ and $\left[K\right]_{ss}$ have dimensions such that each F_i, i=1, 2, ...N, of the vector $\left\{F\right\}$ has the dimensions of force or torque.

In passive systems, as a direct result of Maxwell's Reciprocal Theorem, the following constraints are placed on the coefficients:

$\left[M\right]_{ss}$ and $\left[K\right]_{ss}$ are null matrices.

$\left[C\right]$ is a non negative definite symmetric matrix.

$\left[M\right]_p + \left[M\right]_p^T$ a symmetric matrix, is non negative definite.

$\left[K\right]_p + \left[K\right]_p^T$ a symmetric matrix, is non negative definite.

Actually $\left[M\right]_p + \left[M\right]_p^T$ is generally a positive definite matrix.

Applying Lagrange's Equations (From Eq. (1.5)) to the linear system described by Eqs. (1.8) and (1.9)

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L - \frac{\partial}{\partial q_i} L = F_i \qquad i=1, 2, \ldots N \tag{1.11}$$

$$L = T - V = \frac{1}{2}\left\{\dot{q}\right\}^T \left[M\right]_p \left\{\dot{q}\right\} - \frac{1}{2}\left\{q\right\} \left[K\right]_p \left\{q\right\} \tag{1.12}$$

Substituting Eq. (1.12) into Eq. (1.11)

$$\sum_{j=1}^{N} \left[(m_{pij} + m_{pji}) \ddot{q}_j + (k_{pij} + k_{pji}) q_i \right] =$$

$$- \sum_{j=1}^{N} \left[m_{ssij} \ddot{q}_i + c_{ij} \dot{q}_j + k_{ssij} q_j \right] + f_i(t) \qquad i=1, 2, \ldots N \tag{1.13}$$

Rewriting Eq. (1.13) as a vector-matrix equation

$$\left[[M]_p + [M]_p^T + [M]_{ss} \right] \{\ddot{q}\} + [C]\{\dot{q}\}$$

$$+ \left[[K]_p + [K]_p^T + [K]_{ss} \right] \{q\} = \{f(t)\} \tag{1.14}$$

Let

$$[M]_p + [M]_p^T = [M]_s \quad \text{a symmetric matrix}$$

$$[K]_p + [K]_p^T = [K]_s \quad \text{a symmetric matrix}$$

Equation (1.14) may now be rewritten as

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{f(t)\} \tag{1.15}$$

where

$$\begin{aligned} [M] &= [M]_s + [M]_{ss} \\ [K] &= [K]_s + [K]_{ss} \end{aligned} \tag{1.16}$$

As any matrix with constant elements may be decomposed into a symmetric and a skew symmetric matrix there are no restrictions on the elements of $[M]$, $[K]$ and $[C]$ in a general linear dynamic system.

Development of the Normal Mode Theory

Lagrange[2] and Routh[3] confined their attention, for the most part, to passive conservative systems. The equations of motion of such systems may be written as

$$\left[M\right]\left\{\ddot{x}\right\} + \left[K\right]\left\{x\right\} = \left\{f(t)\right\} \tag{1.17}$$

where $\left\{x\right\}$ is a N x 1 column vector, the il element of which, x_i, is the i^{th} generalized coordinate. $\left[M\right]$ and $\left[K\right]$ are symmetric non-negative definite matrices, $\left[M\right]$ generally being positive definite.

Equations of type (1.17) arise in such well known problems as multi degree of freedom mass spring systems, and approximate solutions to the undamped torsional vibrations of crank shafts.

Due to the special form of the matrices in Eq. (1.17), it may be uncoupled in N-space and solved as a superposition of modal solutions. To achieve this, one has to invoke a theorem from matrix algebra which states that "Given two N x N symmetric matrices $\left[A\right]$ and $\left[B\right]$, with $\left[A\right]$ positive definite, there exists a non-singular matrix φ such that

$$\begin{aligned} \left[\varphi\right]^T \left[A\right]\left[\varphi\right] &= \left[I\right] = I, \text{ the identity matrix} \\ \left[\varphi\right]^T \left[B\right]\left[\varphi\right] &= \left[D\right], \text{ a diagonal matrix.''} \end{aligned} \tag{1.18}$$

The diagonal elements of $\left[D\right]$ are the roots of the following polynomial in λ

$$\left\| \left[\lambda\left[A\right] - \left[B\right]\right] \right\| = 0 \tag{1.19}$$

i.e., the eigenvalues of $\left[A\right]^{-1}\left[B\right]$.

The columns of $[\varphi]$ are the eigenvectors of $[A]^{-1}[B]$, normalized so that

$$[\varphi]^T [A][\varphi] = I$$

As Eq. (1.17) is a linear differential equation, superposition holds and in the usual fashion one first solves the homogeneous equation (with $\{f(t)\} \equiv \{0\}$), taking the initial conditions into account, and later the inhomogeneous equation with zero initial conditions.

Solution to the Free Vibration (Homogeneous) Problem

The equations of free vibration may be written as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \tag{1.20}$$

$[M]$ and $[K]$ are symmetric matrices and $[M]$ is a positive definite matrix. Due to the theorem quoted above there exists a transformation $[\varphi]$ such that

$$[\varphi]^T [M][\varphi] = I \tag{1.21}$$

$$[\varphi]^T [K][\varphi] = [\bar{K}], \text{ a diagonal matrix with}$$

diagonal elements equal to the eigenvalues of $[M]^{-1}[K]$.

To solve Eq. (1.20) let

$$\{x\} = [\varphi]\{\eta(t)\} \tag{1.22}$$

Substituting Eq. (1.22) into Eq. (1.20)

$$[M][\varphi]\{\ddot{\eta}(t)\} + [K][\varphi]\{\eta(t)\} = \{0\} \tag{1.23}$$

Premultiply Eq. (1.23) by $[\varphi]^T$

$$[\varphi]^T [M][\varphi]\{\ddot{\eta}(t)\} + [\varphi]^T [K][\varphi]\{\eta(t)\} = \{0\} \tag{1.24}$$

Substituting Eqs. (1.21) into Eq. (1.24)

$$\{\ddot{\eta}(t)\} + [\bar{K}]\{\eta(t)\} = \{0\} \tag{1.25}$$

Equation (1.25) is a set of uncoupled equations of type

$$\ddot{\eta}_i(t) + \bar{K}_{ii}\,\eta_i(t) = 0 \tag{1.26}$$

Each of the equations in Eq. (1.26) is the equation of motion of a single degree of freedom linear oscillator with the well known solution

$$\eta_i(t) = a_i \sin \sqrt{\bar{K}_{ii}}\,t + b_i \cos \sqrt{\bar{K}_{ii}}\,t \tag{1.27}$$

a_i, b_i arbitrary constants i=1, 2, ... N

From Eq. (1.22) and (1.27)

$$\{x\} = [\varphi]\{a_i \sin \sqrt{\bar{K}_{ii}}\,t\} + [\varphi]\{b_i \cos \sqrt{\bar{K}_{ii}}\,t\} \tag{1.28}$$

To determine a_i and b_i, i=1, 2, ... N, the initial conditions are used:

$$\begin{aligned} \{x(t)\}_{t=0} &= \{x(0)\} \\ \{\dot{x}(t)\}_{t=0} &= \{\dot{x}(0)\} \end{aligned} \tag{1.29}$$

From Eqs. (1.28) and (1.29)

$$\begin{aligned} [\varphi]\{b_i\} &= \{x(0)\} \\ \therefore\ \{b_i\} &= [\varphi]^{-1}\{x(0)\} \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} [\varphi]\{\sqrt{\bar{K}_{ii}}\,a_i\} &= \{\dot{x}(0)\} \\ \therefore \{\sqrt{\bar{K}_{ii}}\,a_i\} &= [\varphi]^{-1}\{\dot{x}(0)\} \end{aligned} \tag{1.31}$$

From Eqs. (1.30) and (1.31) the constant vectors $\{a_i\}$ and $\{b_i\}$ may be determined. Knowing a_i and b_i, i=1, 2, ... N, Eq. (1.28) gives the

solution to the free vibration of the undamped system described by Eq. (1. 20).

Solution to the Forced Vibration (Inhomogeneous) Problem

The equations of motion of the forced undamped system may be written as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{f(t)\} \tag{1. 32}$$

As before let

$$\{x\} = [\varphi]\{\eta(t)\} \tag{1. 33}$$

On substituting Eq. (1. 33) into Eq. (1. 32)

$$[M][\varphi]\{\ddot{\eta}(t)\} + [K][\varphi]\{\eta(t)\} = \{f(t)\} \tag{1. 34}$$

Premultiplying Eq. (1. 34) by $[\varphi]^T$ and using Eq. (1. 21)

$$\{\ddot{\eta}(t)\} + [\bar{K}]\{\eta(t)\} = [\varphi]^T \{f(t)\} \tag{1. 35}$$

Let

$$[\varphi]^T \{f(t)\} = \{g(t)\}, \text{ a N x 1 column vector} \tag{1. 36}$$

Using Eq. (1. 36), Eq. (1. 35) may be rewritten

$$\{\ddot{\eta}(t)\} + [\bar{K}]\{\eta(t)\} = \{g(t)\} \tag{1. 37}$$

Equation (1. 37) is a set of uncoupled equations of type

$$\ddot{\eta}_i(t) + \bar{K}_{ii}\,\eta(t) = g_i(t) \qquad i=1, 2, \ldots N \tag{1. 38}$$

Each equation in Eq. (1. 38) is the equation of motion of the forced vibration of a single degree of freedom linear oscillator. The solution to the i^{th} equation is well known, and is given in Duhamel's form

$$\eta_i(t) = \int_0^t h_i(t-\tau)\, g_i(\tau)\, d\tau \tag{1.39}$$

With zero initial conditions, $\eta_i(0) = \dot{\eta}_i(0) = 0$

$$h_i(\tau) = \frac{1}{\sqrt{\bar{K}_{ii}}} \sin \sqrt{\bar{K}_{ii}}\,\tau \tag{1.40}$$

where $h_i(\tau)$, the fundamental solution, is the response of the uncoupled oscillator to a delta function at t=0 starting with zero initial conditions. From Eqs. (1.33), (1.39) and (1.40) the solution to the forced vibration of the undamped system is given by

$$\{x\} = [\psi]\left\{\int_0^t \frac{1}{\sqrt{\bar{K}_{ii}}} \sin \sqrt{\bar{K}_{ii}}\,(t-\tau)\, g_i(\tau)\, d\tau\right\} \tag{1.41}$$

From Eq. (1.41) it is easy to show that in this case

$$\{x(0)\} = \{\dot{x}(0)\} = \{0\}$$

Therefore, to obtain the solution to the forced vibration problem with non-zero initial conditions, $\{x(0)\}$ and $\{\dot{x}(0)\}$, one has to superimpose the solution to the homogeneous problem (Eq. (1.28)) on the solution to the inhomogeneous problem (Eq. (1.41)).

Rayleigh's Work with Damped Systems

Rayleigh[4] introduced the concept of the dissipation function which, in the development given above, gives rise to the term $\sum_{j=1}^{N} C_{ij}\dot{q}_j$ in the generalized force F_i. For passive systems which include the dissipation function, the equations of motion may be written as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{1.42}$$

Here, the matrix $[C]$ is symmetric and $[C]\{\dot{x}\}$ is the linear viscous damping term of the system. Rayleigh foresaw the difficulty of trying to solve Eq. (1.42) by the then available modal methods. Unless the system can be uncoupled in N-space the modal scheme of solution, given above, breaks down. To ensure the uncoupling in N-space Rayleigh worked with systems which are now said to possess Rayleigh type damping, i.e., systems such that

$$[C] = \alpha[M] + \beta[K] \tag{1.43}$$

α , β arbitrary constants

As $[M]$ and $[K]$ can be diagonalized simultaneously by $[\varphi]$, so also can $[C]$ for

$$\text{Given} \quad [\varphi]^T[M][\varphi] = I$$

$$\text{and} \quad [\varphi]^T[K][\varphi] = [\bar{K}]$$

$$\text{then} \quad [\varphi]^T[C][\varphi] = [\varphi]^T\left[\alpha[M] + \beta[K]\right][\varphi]$$

$$= \alpha I + \beta[\bar{K}] = [\bar{C}] \text{ , a diagonal matrix} \tag{1.44}$$

Hence, passive systems possessing Rayleigh type damping may be uncoupled in N-space by a simple extension of the method used above on the undamped system.

Outline of the Solution to Passive Rayleigh Damped Systems

Let the system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{1.45}$$

possess Rayleigh type damping $[C] = \alpha [M] + \beta [K]$. To solve Eq. (1.45) let

$$\{x\} = [\varphi]\{\eta(t)\} \tag{1.46}$$

Substituting Eq. (1.46) into Eq. (1.45), premultiplying by $[\varphi]^T$, the system is uncoupled and the i^{th} equation is

$$\ddot{\eta}_i(t) + \bar{C}_{ii}\,\dot{\eta}_i(t) + \bar{K}_{ii}\,\eta_i(t) = q_i(t) \tag{1.47}$$

$$i=1, 2, \ldots, N$$

where

$$\bar{C}_{ii} = \alpha + \beta \bar{K}_{ii}$$

$$\{g_i(t)\} = [\varphi]^T \{f(t)\}\ , \text{ a N x 1 column vector} \tag{1.48}$$

Equation (1.47) may be solved in an analogous fashion to the undamped system (Eq. (1.20)) by first solving the homogeneous problem with the initial conditions, $x(0)$ and $\dot{x}(0)$, and superposing on this solution the response to the forced system with zero initial conditions. The solution to the homogeneous form of Eq. (1.47) is

$$\eta_i(t) = e^{-\frac{\bar{C}_{ii}}{2}t}\left[a_i \sin\sqrt{\bar{K}_{ii} - \left(\frac{\bar{C}_{ii}}{2}\right)^2}\,t + b_i \cos\sqrt{\bar{K}_{ii} - \left(\frac{\bar{C}_{ii}}{2}\right)^2}\,t\right] \tag{1.49}$$

where a_i and b_i are arbitrary constants determined from the initial conditions in a similar manner to that used above for the undamped case. In the solution to the inhomogeneous form of Eq. (1.47) the Duhamel Integral is again used

$$\eta_i(t) = \int_0^t h(t-\tau)\, g_i(\tau)\, d\tau$$

$$\eta_i(0) = \dot{\eta}_i(0) = 0 \qquad (1.50)$$

where

$$h(\tau) = \frac{1}{\sqrt{\bar{K}_{ii} - \left(\frac{\bar{C}_{ii}}{2}\right)^2}}\, e^{-\left(\frac{\bar{C}_{ii}}{2}\right) t} \sin \sqrt{\bar{K}_{ii} - \left(\frac{\bar{C}_{ii}}{2}\right)^2}\, t \qquad (1.51)$$

By using Eqs. (1.46), (1.48), (1.49), (1.50) and (1.51) the general solution to passive systems with Rayleigh type damping may be obtained.

Caughey's Work with Damped Systems

Caughey divided the class of passive damped systems into two mutually exclusive sets.

Classical Systems: Those passive systems which can be uncoupled in N-space and which possess the same normal modes as the undamped system.

Non-Classical Systems: Those passive systems which cannot be uncoupled in N-space and which do not possess the same normal modes as the undamped system.

Note: The designation normal mode comes from the fact that when the systems are uncoupled

$$\begin{aligned} [\varphi]^T [M] [\varphi] &= I \\ [\varphi]^T [K] [\varphi] &= [\bar{K}] \end{aligned} \qquad (1.52)$$

i.e., if $\{\varphi\}_i$ is the i^{th}, i=1, 2, ... N, column vector of the matrix $[\varphi]$

$$\{\varphi\}_j^T [M] \{\varphi\}_i = 0 \qquad i \neq j \tag{1.53}$$

$$\{\varphi\}_j^T [K] \{\varphi\}_i = 0 \qquad i \neq j$$

$$\{\varphi\}_i^T [M] \{\varphi\}_i = 1$$

$$\{\varphi\}_i^T [K] \{\varphi\}_i = K_{ii}$$

Hence the vectors $\{\varphi\}_i$ and $\{\varphi\}_j$ $(i \neq j)$ are normalized with respect to $[M]$.

Caughey showed that a necessary and sufficient condition for a passive system to possess classical normal modes is that the damping matrix $[C]$ be diagonalized by the same transformation which uncouples the undamped system, obtained by setting $[C] = 0$ in the damped system. He, furthermore, derived sufficient conditions for $[C]$ for the system to be classical. Here it will be shown that these conditions are also necessary.

Proof of the Necessity and Sufficiency of Caughey's Series For Passive Systems to be Classically Damped

The Caughey Series

$$[M]^{-1}[C] = \sum_{\ell=0}^{N-1} a_\ell \left([M]^{-1}[K]\right)^\ell \tag{1.54}$$

where a_ℓ are arbitrary constants relates the damping matrix $[C]$ to the inertia and stiffness matrices $[M]$ and $[K]$, respectively. Using some of the ideas of Caughey's original sufficiency proof it will

be shown below that this series is a necessary and sufficient condition on the damping matrix $[C]$ for a passive system to be classical.

In a passive system $[M]$ is symmetric and positive definite, $[K]$ and $[C]$ are symmetric. It is well to point out that the reduction of $[M]$ to an identity matrix, I, instead of to a diagonal matrix $[\bar{M}]$, does not reduce the number of systems which can be uncoupled in N-space and are here called classical systems.

$[M]$, being a symmetric matrix, can be diagonalized by an orthogonal transformation $[\tau]$, i.e.,

$$[\tau]^T [\tau] = I$$

$$[\tau]^T [M][\tau] = [\bar{M}] \text{ , a diagonal matrix}$$

The diagonal elements of $[\bar{M}]$ are the eigenvalues of $[M]$. The transformation matrix $[\tau]$ is not unique if $[M]$ possesses repeated eigenvalues. For, suppose that the first a of the diagonal elements of $[\bar{M}]$ were identical, the matrix

$$[\tau]_1 = [\tau][R] \text{ ,} \tag{1.55}$$

where

$$[R] = \begin{array}{c} \\ a \\ N-a \end{array} \begin{array}{c} \begin{array}{cc} a & N-a \end{array} \\ \left[\begin{array}{c|c} [A] & 0 \\ \hline 0 & I \end{array}\right] \end{array} \text{ - a partitioned matrix}$$

and $[A]$ is any a x a orthogonal matrix (i.e., $[A]^T [A] = I$), will diagonalize $[M]$ (as an orthogonal transformation). As $[M]$ is a positive definite matrix, the diagonal elements of $[\bar{M}]$ are all positive

and so $\left[\bar{M}\right]^{-1/2}$ exists, $\left[\bar{M}\right]^{-1/2}$ is a diagonal matrix whose i^{th} diagonal element is $(\bar{M}_i)^{-1/2}$ where $\bar{M}_i$ is the i^{th} diagonal element of $\left[\bar{M}\right]$. It is therefore possible to reduce $\left[M\right]$ to an identity matrix, I, by a transformation $\left[Q\right] = \left[\tau\right]_1 \left[\bar{M}\right]^{-1/2}$ as follows

$$\left[Q\right]^T \left[M\right] \left[Q\right] = I; \quad \left[Q\right]^T = \left[\bar{M}\right]^{-1/2} \left[\tau\right]_1^T \tag{1.57}$$

Hence if $\left[C\right]$ and $\left[K\right]$ are both diagonalized by $\left[\tau\right]_1$ (as an orthogonal transformation) they are also diagonalized by $\left[Q\right]$ e. g., if

$$\left[\tau\right]_1^T \left[K\right] \left[\tau\right] = \left[\bar{K}\right]_1 \qquad \text{a diagonal matrix}$$

then

$$\left[Q\right]^T \left[K\right] \left[Q\right] = \left[\bar{M}\right]^{-1/2} \left[\tau\right]_1^T \left[K\right] \left[\tau\right] \left[\bar{M}\right]^{-1/2} = \left[\bar{M}\right]^{-1/2} \left[\bar{K}\right] \left[\bar{M}\right]^{-1/2}$$

a diagonal matrix, (1.58)

as the product of diagonal matrices is also a diagonal matrix. Thus the reduction of $\left[M\right]$ to an identity matrix does not decrease the number of passive systems solvable in N-space. Furthermore, it is possible to premultiply the original equation by $\left[M\right]^{-1}$ and proceed to uncouple the system as follows. The equations of motion are

$$\left[M\right] \left\{\ddot{x}\right\} + \left[C\right] \left\{\dot{x}\right\} + \left[K\right] \left\{x\right\} = \left\{f(t)\right\} \tag{1.59}$$

Premultiply Eq. (1.59) by $\left[M\right]^{-1}$

$$\left\{\ddot{x}\right\} + \left[M\right]^{-1} \left[C\right] \left\{\dot{x}\right\} + \left[M\right]^{-1} \left[K\right] \left\{x\right\} = \left[M\right]^{-1} \left\{f(t)\right\} \tag{1.60}$$

Now to uncouple the system it is necessary to diagonalize $\left[M\right]^{-1} \left[C\right]$ and $\left[M\right]^{-1} \left[K\right]$ by the same similarity transformation. It is easy to show that $\left[Q\right]$ is the required similarity transformation, for

$$[Q] = [\tau]_1 [\bar{M}]^{-1/2}; \quad [Q]^{-1} = [\bar{M}]^{-1/2} [\tau]_1^T, \text{ since } [\tau]_1^T = [\tau]_1^{-1}$$

and

$$[\tau]_1^T [C] [\tau]_1 = [\bar{C}]; \quad [\tau]_1^T [K] [\tau]_1 = [\bar{K}] \tag{1.61}$$

$$\therefore [Q]^{-1} ([M]^{-1}[C]) [Q] = [\bar{M}]^{-1/2} [\tau]_1^T [M]^{-1} [\tau]_1 [\tau]_1^T [C] [\tau]_1 [\bar{M}]^{-1/2}$$

$$= [\bar{M}]^{-1/2} [\bar{C}] [\bar{M}]^{-1/2} \quad \text{a diagonal matrix} \tag{1.62}$$

since

$$[\tau]^T [M]^{-1} [\tau] = [\bar{M}]^{-1} = [\bar{M}]^{-1/2} [M]^{-1/2}$$

Likewise

$$[Q]^{-1} ([M]^{-1}[K]) [Q] = [\bar{K}] [\bar{M}]^{-1} \quad \text{a diagonal matrix} \tag{1.63}$$

Hence, it is easy to see from Eq. (1.58) and (1.63) that whether a similarity transformation or an orthogonal transformation is used the uncoupled equations are the same.

Preceeding with the proof of the sufficiency of the Caughey series, it follows from the above that in classical systems, $[M]^{-1}[K]$ and $[M]^{-1}[C]$ must be diagonalizable by the same similarity transformation $[Q]$. To prove the sufficiency of the series one assumes that

$$[Q]^{-1} ([M]^{-1}[K]) [Q] = [\bar{K}]_1, \text{ a diagonal matrix} \tag{1.64}$$

and shows that

$$[M]^{-1}[C] = \sum_{\ell=0}^{N-1} a_\ell ([M]^{-1}[K])^\ell \tag{1.65}$$

is diagonalized by $[Q]$. From Eq. (1.65)

$$[Q]^{-1}\left([M]^{-1}[C]\right)[Q] = \sum_{\ell=0}^{N-1} a_\ell [Q]^{-1}\left([M]^{-1}[K]\right)^\ell [Q] \qquad (1.66)$$

Looking at a typical term of the series on the R. H. S. of Eq. (1. 66) and expanding

$$a_r [Q]^{-1}\left([M]^{-1}[K]\right)^r [Q]$$

$$= a_r \underbrace{[Q]^{-1}[M]^{-1}[K][Q][Q]^{-1}[M]^{-1}[K][Q]\text{----}[Q]^{-1}[M]^{-1}[K][Q]}_{\text{for r factors}}$$

$$= a_r\left([\bar{K}]_1\right)^r \qquad \text{a diagonal matrix} \qquad (1.67)$$

where

$$[\bar{K}]_1 = [Q]^{-1}[M]^{-1}[K][Q]$$

Hence each term on the R. H. S. of Eq. (1. 66) is a diagonal matrix and as the sum of a series of diagonal matrices is itself a diagonal matrix, $[M]^{-1}[C]$ as given by the Caughey series, satisfies the requirements of a classical system. The necessity of the series expansion of $[M]^{-1}[C]$ follows once it is shown that given any arbitrary diagonal matrix $[\bar{C}]$ there exists a unique set of a_ℓ $\ell=0, 1 \ldots N-1$.

From the above

$$[Q]^{-1}\left([M]^{-1}[C]\right)[Q] = [\bar{C}]_1 = \sum_{\ell=0}^{N-1} a_\ell\left([\bar{K}]_1\right)^\ell \qquad (1.68)$$

where now $[\bar{C}]_1$ is a diagonal matrix with specified diagonal terms. Rearranging Eq. (1. 68) as a vector matrix equation

$$\{C\}_1 = [V]\{a\} \qquad (1.69)$$

where $\{C\}_1$ is a N x 1 vector, the i^{th} element of which is the i^{th}

diagonal element of $\left[C\right]_1$. $\left\{a\right\}$ is a N x 1 vector, the i^{th} element of which is a_{i-1}

$$\left[V\right] = \begin{bmatrix} 1 & \bar{K}_{1_{11}} & \bar{K}_{1_{11}}^2 & \bar{K}_{1_{11}}^3 & \cdots & \bar{K}_{1_{11}}^{N-1} \\ 1 & \bar{K}_{1_{22}} & \bar{K}_{1_{22}}^2 & \bar{K}_{1_{22}}^3 & \cdots & \bar{K}_{1_{22}}^{N-1} \\ \vdots & & & & & \\ 1 & \bar{K}_{1_{NN}} & \bar{K}_{1_{NN}}^2 & \bar{K}_{1_{NN}}^3 & \cdots & \bar{K}_{1_{NN}}^{N-1} \end{bmatrix}, \text{ a N x N matrix} \tag{1.70}$$

where $\bar{K}_{1_{ii}}$ is the i^{th} diagonal term of the diagonal matrix $\left[\bar{K}\right]_1$.

Now $\left\|\left[V\right]\right\|$ is the well known Vandermonde Determinant

$$\left\|\left[V\right]\right\|^2 = \left| \prod_{i,j=1}^{N} (\bar{K}_{1_{ii}} - \bar{K}_{1_{jj}}) \right| \tag{1.71}$$

From Eq. (1.71) $\left[V\right]^{-1}$ exists if and only if there are no two diagonal elements of $\left[\bar{K}\right]_1$, the same. This implies that a unique set of a_ℓ, $\ell = 0, 1 \ldots N-1$ exist for a given vector $\left\{C\right\}_1$, provided $\left[M\right]^{-1}\left[K\right]$ has no repeated eigenvalues. Therefore, the Caughey series expansion of $\left[M\right]^{-1}\left[C\right]$ is a necessary and sufficient condition for passive systems to be classically damped provided $\left[M\right]^{-1}\left[K\right]$ does not possess repeated eigenvalues.

If $\left[M\right]^{-1}\left[K\right]$ has repeated eigenvalues the conditions for the validity of the Caughey series expansion are sufficient though not necessary for the system to be classical. In this case the minimum

polynomial is of degree (N-α), where α is the sum of the multiplicities of the repeated eigenvalues less their number. By the Cayley-Hamilton Theorem the Caughey series may be reduced to an (N-α) term series

$$\left[M\right]^{-1}\left[C\right] = \sum_{\ell=0}^{N-1} a_\ell \left(\left[M\right]^{-1}\left[K\right]\right)^\ell = \sum_{\ell=0}^{N-1-\alpha} b_\ell \left(\left[M\right]^{-1}\left[K\right]\right)^\ell \qquad (1.72)$$

Each b_ℓ, $\ell=0, 1\ldots, N-1-\alpha$ is a combination of a_t's and the coefficients of the minimum polynomial($t=1, 2\ldots N-1$.). Suppose that the first β eigenvalues of $\left[M\right]^{-1}\left[K\right]$ are identical (the position of the repeated eigenvalues on the diagonal of the reduced form of $\left[M\right]^{-1}\left[C\right]$ depends on the position of eigenvectors as columns of $\left[Q\right]$). It is easy to see that the reduced form of $\left[M\right]^{-1}\left[C\right]$, as given by Eq. (1.72) has the following properties:

$$\left[Q\right]^{-1}\left(\left[M\right]^{-1}\left[C\right]\right)\left[Q\right] = \left[\bar{C}\right]_1 \quad , \text{ a diagonal matrix}$$

$$\bar{C}_{1_{ii}},\ i=1\ldots\beta = \sum_{\ell=0}^{N-\beta} b_\ell \left(\bar{K}_{1_{11}}\right)^\ell \qquad (1.73)$$

where $\bar{K}_{1_{11}}$ is the repeated eigenvalue of $\left[M\right]^{-1}\left[K\right]$ of multiplicity β.

$$\bar{C}_{1_{ii}} = \sum_{\ell=0}^{N-\beta} b_\ell \left(\bar{K}_{1_{ii}}\right)^\ell , \quad (i=\beta+1, \beta+2\ldots N) \qquad (1.74)$$

where $\bar{K}_{1_{ii}}$, $i=\beta+1, \beta+2\ldots, N$ is one of the set of distinct eigenvalues.

From Eqs. (1.73) and (1.74), it is possible to uniquely determine the (N-β+1) terms b_ℓ, $\ell=0\ldots N-\beta$, on arbitrarily specifying the (N-β-1) quantities $\bar{C}_{1_{11}}$, $\bar{C}_{1_{ii}}$, $i=\beta+1, \beta+2\ldots N$, of the diagonal matrix

$[\bar{C}]_1$. If a completely arbitrary diagonal matrix $[\bar{C}]_1$, is specified, i.e., with no restrictions on the equality of the first β diagonal terms, as above, one needs to add to the series of Eq. (1.72) a further series derived as shown below.

In the case of repeated eigenvalues of $[M]^{-1}[K]$, the similarity transformation $[Q]$ is not unique, as shown above. If $[Q]$ is a similarity transformation which will diagonalize $[M]^{-1}[K]$, then, $[Q][R]$ will also diagonalize $[M]^{-1}[K]$ where $[R]$ is any partitioned matrix of type

$$[R] = \begin{array}{c} \\ \beta \\ N-\beta \end{array} \left[\begin{array}{c|c} \overset{\beta}{[A]} & \overset{N-\beta}{0} \\ \hline 0 & I \end{array}\right] \quad ; \quad [A] \text{ is any non-singular matrix} \tag{1.75}$$

For convenience let us designate $[Q]^*$ as one similarity transformation that will diagonalize $[M]^{-1}[K]$, then any $[Q] = [Q]^*[R]$ will also diagonalize $[M]^{-1}[K]$ and $[M]^{-1}[C]$, as given by Eq. (1.72).

Consider the matrix product

$$[E_\beta] = [Q]^* \begin{array}{c} \\ \beta \\ N-\beta \end{array} \left[\begin{array}{c|c} \overset{\beta}{[A][\bar{C}]_\alpha [A]} & 0 \\ \hline 0 & \begin{array}{c} 0 \\ \\ N-\beta \end{array} \end{array}\right] \begin{array}{c} \\ \\ N-\beta \end{array} ([Q]^*)^{-1} \tag{1.76}$$

where $[A]$ is any $\beta \times \beta$ non-singular matrix. $[\bar{C}]_\beta$ is a diagonal $\beta \times \beta$ matrix, with diagonal elements

$$\bar{C}_{\beta ii}, \qquad i=1, 2, \ldots, \beta.$$

Applying the similarity transformation $[Q]$ to Eq. (1.76)

$$\left[\Omega\right]^{-1}\left[E_\beta\right]\left[\Omega\right] = \left[R\right]^{-1}\left[\begin{array}{c|c} \left[A\right]\left[\overline{C}\right]_\beta\left[A\right]^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]\left[R\right]$$

$$= \left[\begin{array}{c|c} \left[\overline{C}\right]_\beta & 0 \\ \hline 0 & 0 \end{array}\right], \quad \text{a diagonal matrix} \tag{1.77}$$

Hence, if to Eq. (1.72), $\left[E_\beta\right]$, as defined by Eq. (1.76) is added a necessary and sufficient condition for the form of $\left[M\right]^{-1}\left[C\right]$ is obtained for passive systems when $\left[M\right]^{-1}\left[K\right]$ has β repeated eigenvalues, i.e.,

$$\left[M\right]^{-1}\left[C\right] = \sum_{\ell=0}^{N-\beta} b_\ell \left(\left[M\right]^{-1}\left[K\right]\right)^\ell + \left[\Omega\right]^* \left[\begin{array}{c|c} \left[A\right]\left[\overline{C}\right]_\beta\left[A\right] & 0 \\ \hline 0 & 0 \end{array}\right] \left(\left[\Omega\right]^*\right)^{-1} \tag{1.78}$$

That $\left[M\right]^{-1}\left[C\right]$, as given by Eq. (1.78) may be diagonalized by the same transformation that diagonalizes $\left[M\right]^{-1}\left[K\right]$ follows from the fact that each term of Eq. (1.78) is similar to a diagonal matrix (using transformation $\left[\Omega\right]$). The necessity of the condition on $\left[M\right]^{-1}\left[C\right]$ results from the fact that any diagonal matrix, $\left[\overline{C}\right]_1$, of order N may be partitioned

$$\left[\overline{C}\right]_1 = \begin{array}{c} \\ \beta \\ N-\beta \end{array} \overset{\beta}{\left[\begin{array}{c|c} \overline{C}_{\beta} + \overline{C}_{1_{11}} I_\beta & 0 \\ \hline 0 & \overline{C}^{\,1}_{N-\beta} \end{array}\right]} \tag{1.79}$$

Where I_β is a β x β identity matrix, $\left[\overline{C}\right]^1_{N-\beta}$ is a $(N-\beta)$ diagonal matrix with i^{th} diagonal term

$$\overline{C}_{1_{i+\beta,\, i+\beta}}, \qquad i=1, 2, \ldots, N-\beta$$

It was shown above that given any set of $\bar{C}_{1_{i+\beta,\, i+\beta}}$, $i=1, 2 \ldots, N-\beta$ and $\bar{C}_{1_{11}}$ there exists a unique set of b_ℓ, $\ell=0, 1 \ldots N-1-\beta$ to contribute the first term of Eq. (1. 78). The second term is obtained once $\left[\bar{C}\right]_\beta$ and $\left[A\right]$ are specified.

Necessary and Sufficient Conditions for Classical Passive Systems

Although the series developed above are useful for synthesis the question of an operational necessary and sufficient condition for a passive system to be classical still arises. It will now be shown that given a passive system with matrixes $\left[M\right]$, $\left[K\right]$ and $\left[C\right]$ the system is classical if and only if $\left[M\right]^{-1}\left[K\right]$ and $\left[M\right]^{-1}\left[C\right]$ commute, i. e.,

$$\left[M\right]^{-1}\left[K\right]\left[M\right]^{-1}\left[C\right] = \left[M\right]^{-1}\left[C\right]\left[M\right]^{-1}\left[K\right] \tag{1. 80}$$

To show the necessity of Eq. (1. 80), from Eqs. (1. 62) and (1. 63) the passive system is classical if a similarity transformation $\left[Q\right]$ exists such that

$$\left[Q\right]^{-1}\left(\left[M\right]^{-1}\left[K\right]\right)\left[Q\right] = \left[\bar{K}\right]_1, \text{ a diagonal matrix of order N} \tag{1. 81}$$

and

$$\left[Q\right]^{-1}\left(\left[M\right]^{-1}\left[C\right]\right)\left[Q\right] = \left[\bar{C}\right]_1, \text{ a diagonal matrix of order N} \tag{1. 82}$$

From Eqs. (1. 81) and (1. 82)

$$\left[Q\right]^{-1}\left(\left[M\right]^{-1}\left[K\right]\right)\left[Q\right]\left[Q\right]^{-1}\left(\left[M\right]^{-1}\left[C\right]\right)\left[Q\right] = \left[\bar{K}\right]_1\left[\bar{C}\right]_1 \tag{1. 83}$$

$$\therefore \left[M\right]^{-1}\left[K\right]\left[M\right]^{-1}\left[C\right] = \left[Q\right]\left[\bar{K}\right]_1\left[\bar{C}\right]_1\left[Q\right]^{-1} \tag{1. 84}$$

likewise

$$\left[M\right]^{-1}\left[C\right]\left[M\right]^{-1}\left[K\right] = \left[Q\right]\left[\bar{C}\right]_1\left[\bar{K}\right]_1\left[Q\right]^{-1} \tag{1. 85}$$

As $[\bar{C}]_1$, and $[\bar{K}]_1$, are both diagonal matrices

$$[\bar{C}]_1 [\bar{K}]_{\#1} = [\bar{K}]_1 [\bar{C}]_1 \tag{1.86}$$

From Eqs. (1.83), (1.84), and (1.85)

$$[M]^{-1}[K][M]^{-1}[C] = [M]^{-1}[C][M]^{-1}[K] \tag{1.87}$$

which proves the necessity of the condition.

To show the sufficiency of Eq. (1.80) for the system to be classical, as $[M]$ and $[K]$ are symmetric and $[M]$ is positive definite there exists a similarity transformation $[Q]$ such that

$$[Q]^{-1}([M]^{-1}[K])[Q] = [\bar{K}]_1 \quad \text{, a diagonal matrix of order N} \tag{1.88}$$

From Eq. (1.80)

$$[Q]^{-1}([M]^{-1}[K])[Q][Q]^{-1}([M]^{-1}[C])[Q]$$

$$= [Q]^{-1}([M]^{-1}[C])[Q][Q]^{-1}[M]^{-1}[K][Q] \tag{1.89}$$

Eq. (1.89) may be reduced on substituting Eq. (1.88)

$$[\bar{K}]_1 [Q]^{-1}([M]^{-1}[C])[Q] = [Q]^{-1}([M]^{-1}[C])[Q][\bar{K}]_1 \tag{1.90}$$

Let

$$[Q]^{-1}([M]^{-1}[C])[Q] = [X] \quad \text{a N x N matrix} \tag{1.91}$$

On substituting Eq. (1.91) into Eq. (1.90)

$$[\bar{K}]_1 [X] = [X][\bar{K}]_1 \tag{1.92}$$

Taking the ij^{th} term of both sides

$$\bar{K}_{1ii} X_{ij} = X_{ij} K_{1jj} \tag{1.93}$$

where X_{ij} is the ij^{th} term of $[X]$

$$\therefore X_{ij} = 0, \quad i \neq j \quad \text{if} \quad \bar{K}_{1_{kk}} \neq \bar{K}_{1_{jj}}, \quad i \neq j \tag{1.94}$$

Hence, if the eigenvalues of $[M]^{-1}[K]$ are distinct (i.e., $\bar{K}_{1_{ii}} \neq \bar{K}_{1_{jj}}$ $i \neq j$), $[X]$ is a diagonal matrix and the condition given by Eq. (1.80) is sufficient for passive systems to be classical. If the eigenvalues of $[M]^{-1}[K]$ are repeated then the matrix $[X]$ may be partitioned as follows

$$[X] = \left[\begin{array}{c|c} [X]_a & 0 \\ \hline 0 & [\bar{X}]_{N-a} \end{array}\right] \tag{1.95}$$

where $[X]_a$ is a a x a matrix.

$[\bar{X}]_{N-a}$ is a $(N-a)$ diagonal matrix corresponding to the $(N-a)$ distinct eigenvalues of $[M]^{-1}[K]$. As $[M]$ and $[C]$ are symmetric and $[M]$ is positive definite the canonical form of $[M]^{-1}[C]$ is a diagonal matrix. Therefore there exists a similarity transformation $[R]$, where

$$[R] = \overset{a}{\left[\begin{array}{c|c} [A] & 0 \\ \hline 0 & I \end{array}\right]}\ a, \tag{1.96}$$

such that $[R]^{-1}[X][R]$ is a strictly diagonal matrix.

$$[R]^{-1}[X][R] = \left[\begin{array}{c|c} [A]^{-1}[X]_a[A] & 0 \\ \hline 0 & I \end{array}\right]$$

i.e., $[A]^{-1}[X]_a[A]$ is a a x a diagonal matrix. But in this case, with a repeated eigenvalues of $[M]^{-1}[K]$,

$$[R]^{-1}[Q]^{-1}([M]^{-1}[K])[Q][R] = [\bar{K}]_1 \quad , \text{ a diagonal}$$

matrix with the first a diagonal elements identical.

Thus the necessity and sufficiency of Eq. (1. 80) for passive systems to be classical has been proved.

Foss's Work With Passive Non Classical System

Although Foss[6] did not, as Caughey had done, derive necessary and sufficient conditions on the damping matrix $[C]$ for solvability in N-space, he did realize that not all passive systems can be solved in N-space. Using a method well known to mathematicians, but first applied to vibration problems by Frazer,[15] Duncan and Collar, he formulated the problem as follows:

The equations of motion of passive systems may be written as

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{f(t)\} \tag{1.97}$$

where $[M]$, $[C]$ and $[K]$ are N x N symmetric matrices with $[M]$ positive definite. To Eq. (1.97) add the identity

$$[M]\{\dot{X}\} - [M]\{\dot{X}\} = \{0\} \tag{1.98}$$

to obtain the following set of equations

$$\begin{aligned} [M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} &= \{f(t)\} \\ [M]\{\dot{X}\} - [M]\{\dot{X}\} &= \{0\} \end{aligned} \tag{1.99}$$

This set may be rewritten in 2N-space

$$[R]\{\dot{Z}\} + [S]\{Z\} = \{F(t)\} \tag{1.100}$$

where

$$\left[R\right] = \begin{bmatrix} \left[0\right] & \left[M\right] \\ \left[M\right] & \left[C\right] \end{bmatrix} \quad ; \quad \left[S\right] = \begin{bmatrix} -\left[M\right] & \left[0\right] \\ \left[0\right] & \left[K\right] \end{bmatrix} \tag{1.101}$$

$$\left\{F(t)\right\} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} \quad ; \quad \left\{Z\right\} = \begin{Bmatrix} \dot{X} \\ X \end{Bmatrix}$$

$\left[R\right]$ and $\left[S\right]$ are symmetric 2N x 2N matrices but neither of them are positive definite. $\left\{F(t)\right\}$ and $\left\{Z\right\}$ are 2N x 1 column vectors. Foss assumed that a modal solution to Eq. (1.100) was possible. This, in general, is not true except in the case of distinct eigenvalues of $\left[R\right]^{-1}\left[S\right]$. Basically, the difficulty is that the ordinary eigenvectors of $\left[R\right]^{-1}\left[S\right]$ need not span the 2N space when there are repeated eigenvalues. However, proceeding on Foss's assumption of the existence of a complete set of ordinary eigenvectors,

$$\left[u\right] = \left[R\right]^{-1}\left[S\right] = \left[\begin{array}{c|c} N \left[M\right]^{-1}\left[C\right] & \left[M\right]^{-1}\left[K\right] \\ \hline N \quad -I & 0 \end{array}\right]^{N}_{N}, \text{ a 2N x 2N matrix,} \tag{1.102}$$

may be diagonalized by a similarity transformation $\left[\bar{\Phi}\right]$, the columns of which are the eigenvectors of $\left[R\right]$ and $\left[S\right]$. From the fact that $\left[R\right]$ and $\left[S\right]$ are symmetric

$$\begin{aligned} \left[\bar{\Phi}\right]^T\left[R\right]\left[\bar{\Phi}\right] &= \left[\bar{R}\right] \quad , \text{ a diagonal 2N x 2N matrix} \\ \left[\bar{\Phi}\right]^T\left[S\right]\left[\bar{\Phi}\right] &= \left[\bar{S}\right] \quad , \text{ a diagonal 2N x 2N matrix} \end{aligned} \tag{1.103}$$

Equations (1.103) are the orthogonality conditions in 2N space and may be expanded in terms of N-space quantities. From the form of Eq. (1.100) it is easy to see that the i^{th} column, $\left\{\bar{\Phi}\right\}_i$, of $\left[\bar{\Phi}\right]$ may

be partitioned

$$\{\bar{\Phi}\}_i = \begin{Bmatrix} \alpha_i \{\phi_i\} \\ \{\phi_i\} \end{Bmatrix} \quad i=1, 2\ldots, 2N \tag{1.104}$$

$\{\phi_i\}$ is a N x 1 column vector, α_i is an eigenvalue of $[R]^{-1}[S]$. It is well to note that the $\{\phi_i\}$, i=1, 2, . . . , 2N in general are vectors with complex elements and that the α_i occur as complex conjugate pairs.

To complete the solution of Eq. (1.100), let

$$\{Z\} = [\bar{\Phi}]\{\eta(t)\} \tag{1.105}$$

where $\{\eta(t)\}$ is a 2N x 1 column vector. Substituting Eq. (1.105) into Eq. (1.100),

$$[R][\bar{\Phi}]\{\dot{\eta}(t)\} + [S][\bar{\Phi}]\{\eta(t)\} = \{F(t)\} \tag{1.106}$$

Premultiplying Eq. (1.106) by $[\bar{\Phi}]^T$

$$[\bar{\Phi}]^T[R][\bar{\Phi}]\{\dot{\eta}(t)\} + [\bar{\Phi}]^T[S][\bar{\Phi}]\{\eta(t)\} = [\bar{\Phi}]^T\{F(t)\} \tag{1.107}$$

On substituting Eq. (1.103) into (1.107)

$$[\bar{R}]\{\dot{\eta}(t)\} + [\bar{S}]\{\eta(t)\} = \{G(t)\} \tag{1.108}$$

where $\{G(t)\} = [\bar{\Phi}]^T\{F(t)\}$ a 2N x 1 column vector. Equation (1.108) is a set of uncoupled equations of type

$$\bar{R}_{ii}\ \dot{\eta}_i(t) + \bar{S}_{ii}\ \eta_i(t) = G_i(t) \tag{1.109}$$

$$i=1, 2, \ldots, 2N$$

with general solution

$$\eta_i(t) = \frac{1}{\bar{R}_{ii}} \int_0^t e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}(t-\tau)} G_i(t)d\tau + \eta_i(0)\, e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}t} \tag{1.110}$$

now

$$\frac{\bar{S}_{ii}}{\bar{R}_{ii}} = a_{ii}\ , \text{ the } i^{th} \text{ eigenvalue of } \left[R\right]^{-1}\left[S\right] \tag{1.111}$$

and

$$\left\{\eta_i(0)\right\} = \left[\Phi\right]^{-1}\left\{Z(0)\right\} \tag{1.112}$$

From Eqs. (1.105), (1.110), (1.111) and (1.112) the general solution of Eq. (1.100) is

$$\left\{Z\right\} = \left[\bar{\Phi}\right]\left\{\eta_i(t)\right\} = \left[\bar{\Phi}\right]\left\{\frac{1}{\bar{R}_{ii}}\int_0^t e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}(t-\tau)} G_i(\tau)\, d\tau\right\} + \left[\bar{\Phi}\right]\left[e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}t}\right]\left[\bar{\Phi}\right]^{-1}\left\{Z(0)\right\} \tag{1.113}$$

where $\left[e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}t}\right]$ is a diagonal matrix with diagonal elements $e^{-\frac{\bar{S}_{ii}}{\bar{R}_{ii}}t}$. From Eq. (1.113) it is seen that the initial condition $\left\{Z(0)\right\}$ influences the solution $\left\{Z\right\}$ in a rather complicated way. However,

$$\left\{X\right\} = \sum_{i=1}^{2N}\left(\frac{1}{R_{ii}}\int_0^t e^{-a_{ii}(t-\tau)} G_i(\tau)d\tau\right)\left\{\phi_i\right\} \tag{1.114}$$

$$\left\{\dot{X}\right\} = \sum_{i=1}^{2N}\left(\frac{1}{R_{ii}}\, a_{ii}\int_0^t e^{-a_{ii}(t-\tau)} G_i(\tau)d\tau\right)\left\{\phi_i\right\} \tag{1.115}$$

provided $\left\{X(0)\right\} = \left\{\dot{X}(0)\right\} = \left\{0\right\}$.

General Theory of Vibration of Damped Linear Systems

As there is essentially no difference in procedure when solving those passive systems not capable of being solved by normal mode methods in either N or 2N space, and the general system characterised by a non symmetric non-singular inertia matrix $[M]$ and non symmetric damping and stiffness matrices $[C]$ and $[K]$ respectively, the following treatment may apply to both of these types of systems. It may also be used for systems solvable in either N or 2N space by the normal mode methods described above.

Foss's 2N formulation does not depend on any properties of the matrices $[M]$, $[K]$ and $[C]$ and therefore it is possible to formulate all linear discrete systems as problems in 2N space.

$$[R]\{\dot{Z}\} + [S]\{Z\} = \{F(t)\} \tag{1.116}$$

where $[R]$ and $[S]$ are as defined above by Eq. (1.101) with no restrictions on the matrices $[M]$, $[K]$ and $[C]$.

$$[R]^{-1} = \left[\begin{array}{c|c} -[M]^{-1}[C][M] & [M]^{-1} \\ \hline [M]^{-1} & [0] \end{array}\right]$$

$[R]^{-1}$ exists if and only if $[M]^{-1}$ exists and in this work attention will be confined to systems with non singular inertia matrices. For a transformation sequence which will transform any system with a singular inertia matrix into a system with a non-singular inertia matrix reference should be made to a report by Dr. Caughey[16] and the present author.

Premultiplying Eq. (1.116) by $[R]^{-1}$ and simplifying, one finds

$$\{\dot{Z}\} + \left[\begin{array}{c|c} [M]^{-1}[C] & [M]^{-1}[K] \\ \hline -I & [0] \end{array}\right] \{Z\} = \left\{\begin{array}{c} [M]^{-1}\{f(t)\} \\ \{0\} \end{array}\right\} \tag{1.117}$$

To simplify the notation, let

$$[A] = \left[\begin{array}{c|c} [M]^{-1}[C] & [M]^{-1}[K] \\ \hline -I & [0] \end{array}\right]; \quad \{N(t)\} = \left\{\begin{array}{c} [M]^{-1}\{f(t)\} \\ \{0\} \end{array}\right\} \tag{1.118}$$

Substituting Eq. (1.118) into Eq. (1.117)

$$\{\dot{Z}\} + [A]\{Z\} = \{N(t)\} \tag{1.119}$$

To solve Eq. (1.119) a theorem from matrix algebra is required: -

"Every square matrix A is reducible to Jordan's canonical form by a similarity transformation $[\tau]$, i.e.,

$[\tau]^{-1}[A][\tau] = [J]$, the Jordan canonical form of $[A]$"

The Jordan canonical form of a matrix with distinct eigenvalues is a strictly diagonal matrix with diagonal elements equal to the eigenvalues. In this case the columns of $[\tau]$ are the eigenvectors of the matrix. Eigenvectors are not uniquely defined in magnitude but for distinct eigenvalues have a unique direction. The Jordan form for a matrix with repeated eigenvalues may or may not be diagonal. In either case the eigenvector associated with a repeated root is unique neither in magnitude nor direction. In the case where the Jordan form is a diagonal matrix the diagonal elements are the eigenvalues and the columns of $[\tau]$ are eigenvectors of the matrix. The non-diagonal Jordan form arises whenever there are less ordinary eigenvectors associated with a repeated eigenvalue than the multiplicity of the

eigenvalue. For symmetric matrices it is always possible to obtain the same number of eigenvectors, associated with a repeated eigenvalue, as the multiplicity of the eigenvalue. The Jordan form of a matrix with less eigenvectors than the order of the matrix, is a matrix of the same order, with diagonal elements equal to the eigenvalues, the elements immediately above and parallel to the diagonal are either equal to 1 or 0 and all other elements are zero. The elements, off the diagonal, equal to 1 can occur only when the two nearest diagonal elements are identical. A few examples of the Jordan form of matrices are given below:

$$[A] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -1 & 1 \end{bmatrix} \quad \| [A - \lambda I] \| = (\lambda - 1)(\lambda - 2)(\lambda - 3) \tag{1.120}$$

i.e., the eigenvalues of $[A]$ are $\lambda = 1, 2, 3$.

$$[\tau]^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \; ; \quad [\tau] = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \tag{1.121}$$

The columns of $[\tau]$ are the eigenvectors of $[A]$

$$[\tau]^{-1} [A] [\tau] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$, the Jordan canonical form of $[A]$.

This is an example of a strictly diagonal Jordan form associated with a matrix which has no repeated eigenvalues.

$$[A] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \| [A - \lambda I] \| = (\lambda - 1)^2 (\lambda - 2)$$

i.e., the eigenvalues of $[A]$ are 1, 1, 2.

The similarity transformation $[\tau]$ to reduce $[A]$ to Jordan form is given by Eq. (1.121)

$$[\tau]^{-1}[A][\tau] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.122}$$

The columns of $[\tau]$ are eigenvectors of $[A]$ and there are two distinct eigenvectors associated with the eigenvalue 1 of multiplicity two.

Finally, two examples of non-diagonal Jordan form matrices are given

$$[A] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \|[A - \lambda I]\| = (\lambda - 2)^3$$

i.e., the eigenvalues of $[A]$ are $\lambda = 2, 2, 2$.

Using the transformation matrix

$$[\tau] \equiv \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} ; \quad [\tau]^{-1}[A][\tau] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.123}$$

Associated with the eigenvalue 2 of multiplicity 3 there is only one ordinary eigenvector

$$\{\tau_1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{1.124}$$

The other columns of $[\tau]$ are generalized eigenvectors of rank 2 and 3 respectively. A vector $\{X_K\}$ for which

$$\begin{aligned} \left[[A] - \lambda_o I\right]^{K-1} \{X_K\} &\neq 0 \\ \left[[A] - \lambda_o I\right]^{K} \{X_K\} &= 0 \end{aligned} \tag{1.125}$$

is called a generalized eigenvector of rank K corresponding to the eigenvalue λ_o. It is easy to show that generalized eigenvectors of different rank are linearly independent and that

$\left[\left[A\right] - \lambda_o I\right]^j \left\{X_K\right\} = \left\{X_{K-j}\right\}$ a generalized eigenvector of rank K-j $(j \leq K+1)$, corresponding to the eigenvalue λ_o. A generalized eigenvector of rank 1 is an ordinary eigenvector. There are various techniques[17, 18, 19] available for determining the generalized eigenvectors of a matrix but these will not be discussed here. Due to Eq. (1.125) the generalized eigenvectors occur in chains and this is repeated in the form of series of elements equal to 1 in the off diagonal terms, e.g.,

$$\left[A\right]^{(17)} = \begin{bmatrix} 5 & -1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & -1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}; \quad \begin{matrix} \text{rank} \\ \\ \\ \\ \\ \\ \text{Eigenvalue} \end{matrix} \begin{matrix} 1 & 2 & 3 & 1 & 2 & 1 \end{matrix}$$

$$\left[\tau\right] = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} 4 & 4 & 4 & 4 & 4 & 2 \end{matrix} \qquad (1.126)$$

$\left\|\left[\left[A\right] - \lambda I\right]\right\| = (\lambda - 4)^5 (\lambda - 2)$, i.e., 4 is an eigenvalue of multiplicity 5 and 2 is an eigenvalue of multiplicity 1.

$$\left[\tau\right]^{-1}\left[A\right]\left[\tau\right] = \left[\begin{array}{ccc|ccc} 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right], \text{ the Jordan canonical form of } \left[A\right] \qquad (1.127)$$

From Eqs. (1.120) and (1.127) it is seen that the position of the off diagonal terms, equal to 1, depends on the chains of generalized eigenvectors.

To complete the solution of Eq. (1.119), let

$$\{z\} = [\tau]\{\eta(t)\} \tag{1.128}$$

Substitute Eq. (1.128) into Eq. (1.119) and premultiply by $[\tau]^{-1}$

$$[\tau]^{-1}[\tau]\{\dot{\eta}(t)\} + [\tau]^{-1}[A][\tau] = [\tau]^{-1}\{N(t)\} \tag{1.129}$$

Equation (1.129) may be simplified

$$\{\dot{\eta}(t)\} + [J]\{\eta(t)\} = \{\mathcal{N}(t)\}$$

where $[J] = [\tau]^{-1}[A][\tau]$ is the Jordan canonical form of $[A]$

and

$$\{\mathcal{N}(t)\} = [\tau]^{-1}\{N(t)\}\text{, a 2N x 1 column vector} \tag{1.130}$$

If $[J]$ is a strictly diagonal matrix, Eq. (1.130) is a set of completely uncoupled equations with the i^{th} equation

$$\dot{\eta}_i(t) + J_{ii}\,\eta_i(t) = \mathcal{N}_i(t) \tag{1.131}$$

This system may therefore be solved by exactly the same procedure as was used on the solution of passive systems by Foss's method in 2N-space. If $[J]$ is not a strictly diagonal matrix then it is still possible to solve Eq. (1.130) by first solving the 2N-th equation and proceeding upwards until the complete set is solved. The 2N-th equation is always completely uncoupled and has solution

$$\eta_{2N}(t) = \int_0^t e^{-J_{2N2N}\tau}\mathcal{N}_{2N}(t-\tau)d\tau + \eta_{2N}(0)\,e^{-J_{2N2N}t} \tag{1.132}$$

where J_{2N2N} is the 2N-th diagonal element of $[J]$. Proceeding upwards through the (2N-1)-th (2N-2)-th --- equation until a coupling term is reached at the i^{th} equation. The i^{th} and the (i+1)-th

equations may be written (in general) as

$$\dot{\eta}_i(t) + J_{ii}\,\eta_i(t) + \eta_{i+1}(t) = \mathcal{J}_i(t) \tag{1.133}$$

$$\dot{\eta}_{i+1}(t) + J_{i+1\,i+1}\,\eta_{i+1}(t) = \mathcal{J}_{i+1}(t) \tag{1.134}$$

From the form of $\left[J\right]$,

$$J_{i+1i+1} = J_{ii}$$

$$\therefore\ \eta_{i+1}(t) = \int_0^t e^{-J_{ii}\tau}\,\mathcal{J}_{i+1}(t-\tau)\,d\tau + \eta_{i+1}(0)\,e^{-J_{ii}t} \tag{1.135}$$

On substituting Eq. (1.135) into Eq. (1.133)

$$\dot{\eta}_i(t) + J_{ii}\,\eta_i(t) = \mathcal{J}_i(t) - \left\{\int_0^t e^{-J_{ii}\tau}\,\mathcal{J}_{i+1}(t-\tau)\,d\tau + \eta_{i+1}(0)\,e^{-J_{ii}t}\right\} \tag{1.136}$$

Solving (1.136)

$$\eta_i(t) = \int_0^t e^{-J_{ii}\tau}\,\mathcal{J}_i(t-\tau)\,d\tau - \int_0^t e^{-J_{ii}(t-\tau)}\left\{\int_0^\tau e^{-J_{ii}(\tau-\eta)}\,\mathcal{J}_{i+1}(\eta)\,d\eta + \eta_{i+1}(0)\,e^{-J_{ii}\tau}\right\}d\tau + \eta_i(0)\,e^{-J_{ii}t}$$

$$= \int_0^t e^{-J_{ii}\tau}\,\mathcal{J}_i(t-\tau)\,dr + t\eta_{i+1}(0)\,e^{-J_{ii}t} + \eta_i(0)\,e^{-J_{ii}t}$$

$$+ \int_0^t \left\{\int_0^\tau e^{-J_{ii}(t-\eta)}\,\mathcal{J}_{i+1}(\eta)\,d\eta\right\}d\tau\,.$$

But

$$\int_0^t \left\{ \int_0^{\tau} e^{-J_{ii}(t-\eta)} \mathcal{Y}_{i+1}(\eta)\, d\eta \right\} d\tau = \int_0^t (t-\tau)\, e^{-J_{ii}(t-\tau)} \mathcal{Y}_{i+1}(\tau)\, d\tau$$

$$\therefore \eta_i(t) = \int_0^t e^{-J_{ii}\tau} \mathcal{Y}_i(t-\tau)\, d\tau + \int_0^t (t-\tau)\, e^{-J_{ii}(t-\tau)} \mathcal{Y}_{i+1}(\tau)\, d\tau$$

$$+ \eta_i(0)\, e^{-J_{ii}t} + t\, \eta_{i+1}(0)\, e^{-J_{ii}t} \qquad (1.137)$$

Proceeding upwards, in this manner, until the first equation is solved the solution to the original system may be obtained by substituting

$$\begin{aligned} [Z] &= [\tau]\{\eta(t)\} \\ \{\eta(0)\} &= [\tau]^{-1}\{Z(0)\} \end{aligned} \qquad (1.138)$$

Discussion on Classes of Linear Systems

Although the general theory can be used to solve any discrete linear system, it is to the advantage of the analyst to use the usual normal mode approach in N-space or Foss's formulation in 2N-space for those problems which can be solved by these methods. In the definition of Classical Systems, Caughey assumed that the system was passive and hence that the matrices $[M]$, $[K]$ and $[C]$ were symmetric. As the Jordan canonical form of a symmetric matrix is strictly diagonal many simplifications result from dealing with passive systems. However, there are many non symmetric systems which are solvable in N space by similarity transformation and now

a brief discussion of these systems is given.

From the above work, it is easy to see that the essential criterion of solvability in N space is that given the three matrices $[M]$, $[K]$ and $[C]$ there exists a similarity transformation $[\mathcal{T}]$ such that

$$\begin{aligned} [\mathcal{T}]^{-1}[M][\mathcal{T}] &= [M]_J \text{ the Jordan canonical form of } [M] \\ [\mathcal{T}]^{-1}[K][\mathcal{T}] &= [K]_J \text{ the Jordan canonical form of } [K] \\ [\mathcal{T}]^{-1}[C][\mathcal{T}] &= [C]_J \text{ the Jordan canonical form of } [C] \end{aligned} \tag{1.139}$$

for if $[\mathcal{T}]$ exists, then independent of the precise form of $[M]_J$, $[K]_J$ and $[C]_J$ (i.e., diagonal or non-diagonal) the system may be solved in a manner similar to that shown above in the general 2N formulation. Confining attention for the moment to those systems in which $[M]_J$, $[K]_J$ and $[C]_J$ are strictly diagonal matrices and in which $[M]^{-1}$ exists (actually, $[M]$ is generally symmetric and positive definite) it is easy to see that Eq. (1.139) reduces to the existence of $[\mathcal{T}]$ such that

$$\begin{aligned} [\mathcal{T}]^{-1}([M]^{-1}[C])\mathcal{T} &= [\bar{C}]_1 \text{ a diagonal matrix} \\ [\mathcal{T}]^{-1}([M]^{-1}[K])\mathcal{T} &= [\bar{K}]_1 \text{ a diagonal matrix} \end{aligned} \tag{1.140}$$

If $[M]^{-1}$ and $[C]$ are diagonalizable by the same similarity transformation $[\mathcal{T}]$ so is $[M]^{-1}[C]$ but the converse does not hold as is easily seen from the following example

$$[M] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad [C] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad [M]^{-1}[C] = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \tag{1.141}$$

As $[M]^{-1}[C]$ is normal, i.e., $[M]^{-1}[C]([M]^{-1}[C])^T = ([M]^{-1}[C])^T [M]^{-1}[C]$ it can be diagonalized by a similarity transformation. $[M]$ is symmetric and so can be diagonalized but as $[C]$ is in Jordan canonical form it cannot be further reduced.

There are many necessary and sufficient conditions for a matrix $[A]$ to be similar to a diagonal matrix, including, for example, (i) $[A]$ is non defective (elementary divisors are linear), (ii) minimum polynomial of $[A]$ has distinct factors and (iii) certain conditions on the principal idempotents of $[A]$ are obeyed. However, most of the well known necessary and sufficient conditions are not in a readily usable form. The easiest sufficient condition to check is that $[A]$ be normal, i.e., $[A][A]^T = [A]^T[A]$. If $[A]$ is separated into its symmetric and skew-symmetric parts $[A]_s$ and $[A]_{ss}$ respectively, this condition reduces to

$$[A]_s [A]_{ss} = [A]_{ss} [A]_s$$

From the proof given in the discussion of classical systems, it is seen that, for general $[M]$, $[K]$ and $[C]$ matrices, the Caughey Series with additions due to repeated roots Eq. (1.76) is still a necessary and sufficient condition on $[C]$ for system diagonizability in N-space provided $[M]^{-1}[K]$ is diagonalizable. Furthermore, if $[M]^{-1}[K]$ is diagonalizable the condition of commutability of $[M]^{-1}[K]$ and $[M]^{-1}[C]$ is now only a necessary condition for system diagonalizability in N space. This follows, in the case when $[M]^{-1}[K]$ has repeated eigenvalues, from the fact that the Jordan canonical form of $[X]_\alpha$, - the $\alpha \times \alpha$ matrix of Eq. (1.95), may not be

reducible to strictly diagonal form as it was in the case of classical systems.

In many systems $[M]$ is symmetric and positive definite and $[K]$ and $[C]$ are both normal matrices. In this case it is necessary and sufficient for $[M]^{-1}[K]$ and $[M]^{-1}[C]$ to commute for system diagonalizability in N-space. It is possible for systems to possess matrices $[M]$, $[K]$ and $[C]$ such that the same similarity transformation will reduce each matrix to either a diagonal or non-diagonal Jordan form. Such systems involving non-diagonal Jordan forms in N-space are not easily recognized and are best solved by the general theory in 2N-space. It can be shown that systems with non-diagonal Jordan forms in N-space also have non-diagonal Jordan forms in 2N-space.

Excitation of Pure Modes

In a previous work[7] by the author it was shown that a pure mode of vibration may be excited in classical systems by either initial conditions or by a distribution of forces. Caughey[5] first recognized that classical systems have real eigenvectors and that, when vibrating in a pure mode, all the masses pass through their equilibrium points simultaneously. It was further shown that for passive systems solvable only in 2N-space a pure mode could be excited by initial conditions , but not force excited. From the general formulation in 2N-space it will now be shown that it is always possible to select initial conditions so that the system will vibrate in any pure mode associated with an ordinary eigenvector.

The equations of motion in 2N-space may be written as

$$I\{\dot{Z}\} + [A]\{Z\} = \{\mathcal{Y}(t)\}$$

where

$$\{Z\} = \begin{Bmatrix} \dot{X} \\ X \end{Bmatrix}; \{\mathcal{Y}(t)\} = \begin{Bmatrix} [M]^{-1}\{f(t)\} \\ \{0\} \end{Bmatrix}; [A] = \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -I & 0 \end{bmatrix} \tag{1.142}$$

If the system is not force excited $\{\mathcal{Y}(t)\} = \{0\}$ and, as shown above Eq. (1.142), may be solved as follows. Let

$$\{Z\} = [\mathcal{T}]\{\eta(t)\} \tag{1.143}$$

where $[\mathcal{T}]^{-1}[A][\mathcal{T}] = [J]$ the Jordan form of $[A]$. On substituting Eq. (1.143) into Eq. (1.142) and premultiplying by $[\mathcal{T}]^{-1}$

$$\{\dot{\eta}(t)\} + [J]\{\eta(t)\} = \{0\} \tag{1.144}$$

$$\{\eta(0)\} = [\mathcal{T}]^{-1}\{Z(0)\} \tag{1.145}$$

If $[J]$ is a strictly diagonal matrix with diagonal elements J_{ii} i=1, 2, . . . , 2N. The solution to Eq. (1.144) has the form of

$$\eta_i(t) = \eta_i(0)\, e^{-J_{ii}t}, \quad i=1, 2, \ldots, 2N \tag{1.146}$$

Hence, from Eq. (1.143) it is seen that the system will vibrate in the i^{th} mode if $\eta_i(0) \neq 0$ $\eta_j(0) = 0$, j=1, 2, . . . , 2N ≠ i, i.e., if

$$\{Z(0)\} = \eta_i(0)\{\mathcal{T}_i\} \tag{1.147}$$

where $\{\mathcal{T}_i\}$ is the i^{th} row of $[\mathcal{T}]$ or the i^{th} eigenvector of $[A]$. Now if $\{\mathcal{T}_i\}$ is a vector with essentially complex elements (i.e., it cannot be written as $\{\mathcal{T}_i\} = a_i\{X_i\}$ where a_i is complex and $\{X\}$, a 2N x 1 vector with all real elements) $\{Z(0)\}$ as given by Eq. (1.147) is not real

and is not an admissible set of zero conditions. However, in this case there exists another column of $[\mathcal{Z}]$, $\{\mathcal{Z}_{i+N}\}$ such that

$$\{\overline{\mathcal{Z}}_i\} = \{\mathcal{Z}_{i+N}\} \tag{1.148}$$

where — denotes taking the complex conjugate of each element of $\{\mathcal{Z}_i\}$. Actually due to the real matrices of the system it can be shown that essentially complex vectors are only associated with complex roots and that the complex roots occur as complex conjugate pairs and hence that essentially complex vectors must occur in pairs and must satisfy Eq. (1.148). From these remarks it can be seen that if

$$\{Z(0)\} = \eta_i(0)\{\mathcal{Z}_i\} + \overline{\eta_i(0)}\{\mathcal{Z}_{i+N}\} \tag{1.149}$$

the i^{th} mode of vibration is excited provided J_{ii} and $J_{i+N\ i+N}$ are complex conjugate pairs. If J_{ii} is real then the initial conditions of the form of Eq. (1.147) are real and will excite the i^{th} mode. In the case of repeated roots and a complete set of eigenvectors there are no special difficulties except that the mode shapes are not unique, i.e., if $\{\mathcal{Z}_1\}, \{\mathcal{Z}_2\} \cdots \{\mathcal{Z}_a\}$ are a set of eigenvectors associated with an a-fold eigenvalue any linear combination of these eigenvectors is also an eigenvector and the system may be excited in a pure mode by suitable initial conditions of type Eq. (1.147) or Eq. (1.148) depending on whether the eigenvalue is real or complex. But, in this case the mode shape is not unique.

In the case of repeated eigenvalues of $[A]$ without a complete set of ordinary eigenvectors it is easy to see that it is not possible to

excite the system by initial conditions to vibrate in a mode that corresponds to a generalized eigenvector. For, if the j^{th} column of $[\tau]$ is a generalized eigenvector of $[A]$ of rank r, $\eta_j(t)$ must be non zero to have the system vibrating in this mode but as the $(j-1)j$-th element of $[J]$ is 1 the $(j-1)^{st}$ equation is coupled with the j^{th}, i. e.,

$$\eta_{j-1}(t) = \eta_{j-1}(0)\, e^{-J_i t} + \int_0^t \eta_j(\tau)\, e^{-J_j(t-\tau)}\, d\tau \qquad (1.150)$$

where $\eta_j(t)$, the response of the j^{th} equation, depends on $\eta_j(0)$ and if the j^{th} equation is forced through the coupling of the j^{th} equation with the $(j+1)^{th}$ equation. Due to the fact that $\eta_j(t)$ must have a term of type $t^s e^{-J_j t}$ the integral term in Eq. (1.150) must contain a term of $t^{s+1} e^{-J_j t}$. $0 \leq s < r-1$ Hence, it is impossible, in general, to select a set of initial conditions to excite the system to vibrate in any pure mode corresponding to a generalized eigenvector. It is, however, possible to excite the system to vibrate in any mode not associated with a generalized eigenvector by a suitable combination of initial conditions.

Next, the question of forced excitation of pure modes with zero initial conditions will be discussed. Assume for the moment that the i^{th} and $(i+N)^{th}$ (if a complex eigenvalue) mode correspond to ordinary eigenvectors of $[A]$. To force excite this mode, $[\tau]^{-1}\{\mathcal{F}(t)\}$, - a 2N x 1 column vector, must have all zero elements except at the i^{th} and $(i+N)^{th}$ row (if complex eigenvalue) i. e.,

$$[Z]^{-1}\{\dot{y}(t)\} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ R_i(t) \\ 0 \\ \vdots \\ R_{i+N}(t) \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{1.151}$$

On premultiplying Eq. (1.151) by $[Z]$

$$\{\dot{y}(t)\} = R_i\{Z_i\} + R_{i+N}(t)\{Z_{i+N}\} = \begin{Bmatrix} [M]^{-1} & f(t) \\ & 0 \end{Bmatrix} \tag{1.152}$$

On premultiplying Eq. (1.151) by $[S]$ a 2N x 2N matrix

$$[S] = \overset{N}{\left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [M] \end{array}\right]} \quad \text{and noting that} \quad \{Z_i\} = \begin{Bmatrix} \lambda_i\{\phi_i\} \\ \{\phi_i\} \end{Bmatrix}$$

$$R_i(t)\begin{Bmatrix} \lambda_i[M]\{\phi_i\} \\ [M]\{\phi_i\} \end{Bmatrix} + R_{i+N}(t)\begin{Bmatrix} \bar{\lambda}_i[M]\{\bar{\phi}_i\} \\ [M]\{\bar{\phi}_i\} \end{Bmatrix} = \begin{Bmatrix} f(t) \\ \{0\} \end{Bmatrix} \tag{1.153}$$

As λ_i, the eigenvalue of the mode of interest, is complex

$$R_i(t) = \overline{R_{i+N}(t)}$$

if

$$R_i(t)\ \lambda_i[M]\{\phi_i\} + R_{i+N}(t)\ \bar{\lambda}_i[M]\{\bar{\phi}_i\} = \{f(t)\}, \text{ a real vector} \tag{1.154}$$

Let

$$R_i(t) = a_i(t) + \sqrt{-1}\, b_i(t), \qquad \lambda_i = c_i + \sqrt{-1}\, d_i$$

$$\{\phi_i\} = \{R_i\} + \sqrt{-1}\{I_i\}$$

From Eq. (1.153)

$$R_i(t)\,[M]\,\{\phi_i\} + R_{i+N}(t)\,[M]\,\{\bar{\phi}_i\} = \{0\}$$

$$(a_i(t) + \sqrt{-1}\,b_i(t))\left[M\right]\left(\{R_i\} + \sqrt{-1}\{I_i\}\right) + (a_i(t) - \sqrt{-1}\,b_i(t))\left[M\right]\left(\{R_i\} - \sqrt{-1}\{I_i\}\right) = \{0\} \tag{1.155}$$

Equating real and imaginary parts

$$2a_i(t)\,[M]\,\{R_i\} - 2b_i(t)\,[M]\{I_i\} = \{0\} \tag{1.156}$$

$\therefore$ either $\quad a_i(t) = b_i(t) = 0 \tag{1.157}$

or $$\{R_i\} = \frac{b_i(t)}{a_i(t)}\,\{I_i\}\,; \frac{b_i(t)}{a_i(t)} = k, \text{ any real constant} \tag{1.158}$$

Substituting Eq. (1.157) into Eq. (1.153) gives $f(t) \equiv 0$, i.e., no free excitation possible. Equation (1.158) implies that the vector $\{\gamma_i\}$ is an essentially real vector. As classical systems have real eigenvectors this shows that it is possible to force excite a pure mode in classical systems provided Eq. (1.154) is satisfied, i.e.,

$$2\,R\ell\left[\left(R_i(t)\,\lambda_i\right)\,[M]\,\{\phi_i\}\right] = \{f(t)\} \tag{1.159}$$

Knowing λ_i and $\{\phi_i\}$, it is easy to choose $R_i(t)$ and $f(t)$ so that Eq. (1.159) is satisfied. For non classical systems with real eigenvectors and complex eigenvalues it is likewise possible to force excite the system to vibrate in the pure modes corresponding to those real eigenvectors. If the eigenvalue λ_i is real then unless there is another real eigenvalue λ_{i+N} such that $\{\phi_i\} = \{\phi_{i+N}\}$ it is impossible to force excite the system to vibrate in a pure mode. However, in the case where $\{\phi_i\} = \{\phi_{i+N}\}$, which always occurs with real eigenvalues in classical systems, it is possible to force

excite the system in a pure mode.

In the case of generalized eigenvectors it can be shown that due to the coupling between the modes it is not possible to force excite the system to vibrate in a pure mode corresponding to these generalized eigenvectors. This case will not be discussed in detail here as it is easier to prove the statement with the matrix exponential techniques to be developed in Chapter 2.

Computational Methods

The main methods for finding the eigenvalues and eigenvectors of a matrix are Jacob's Method,[19, 20, 21, 22, 23, 24] Given's Modification of Jacob's Method, Power Method and Lanczos' Method. The Jacob and Given's methods work on an annihilation principle and are confined to real symmetric matrices. The Power method is a development of Stodola's method for symmetric matrices and gives good convergence if the matrix is non defective and if the eigenvalue of interest is simple. The Rayleigh quotient can also be used as an estimate of the eigenvalues of a positive definite matrix. Lanczos' method is very well suited to digital computation and it reduces the matrix to a tridiagonal matrix from which the eigenvalues and eigenvectors may be obtained.

There is, then, no lack of methods for determining the eigenvectors and eigenvalues of a matrix. However, if the order of the system is high it is extremely tedious to calculate them by hand calculator. For any system of order higher than 5 some method of

automatic calculation is required to facilitate the calculations.

The great virtue of the modal method of solution is the physical insight it gives into the synthesis of the system. However, it must be realized that calculating the eigenvalues and eigenvectors of a matrix may not be the most efficient method of determining the response of the system. In the next chapter a presentation based on the matrix exponential is given together with a simple algorithm for numerical calculation.

In the past it has been thought that integral transform methods had to be used if the traditional modal approach was not applicable. In point of fact the use of integral transform methods results in the same type of problems as when non-diagonal Jordan forms arise in the above work. Indeed, the main reason that the integral transform techniques were successful was that, due to the order of the systems solved, it was possible to invert the literal matrices of the problem. If the order of the problem is increased the inversion of the literal matrices would be a formidable task which must be done by hand as a computer cannot be programmed to do it.

CHAPTER 2

FURTHER RESULTS FOR DISCRETE SYSTEMS

Introduction

In this chapter, the discussion of discrete systems is continued. First, a new formulation of the general theory of vibration of these systems is given. This formulation using the ideas of matrix exponentials, has the virtue of elegance and compactness besides being in a form very suitable for digital computation. The question of the stability of general systems is discussed and some new results, obtained by using Liapunov's Second Method are given. A section on perturbation analysis is included to facilitate computation of systems solvable to within terms of small order in N space. Finally, the methods available for bounding the eigenvalues of a matrix are reviewed and applied to the problem under discussion.

Theory

The Matrix Exponential Formulation of the General Theory

Definition of a function of a matrix:[11, 25, 31] Let A be a N x N matrix with k ($N \geq k$) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Repeated eigenvalues are only counted once in this notation. Let f be an analytic function in the open set containing $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let $p(\lambda)$ be a polynomial such that

$$p^{\ell}(\lambda_j) = f^{\ell}(\lambda_j) \qquad \ell = 0, 1, \ldots, r_j - 1, \quad j = 1, 2, \ldots, k$$

where superscript ℓ indicates differentiation and r_j is the highest rank of any generalized eigenvector associated with the repeated eigenvalue

λ_j. If λ_j is a simple eigenvalue of the matrix $[A]$, $r_j = 1$. Then the function $f[A]$ of the matrix $[A]$ is defined as

$$f[A] = p[A].$$

As $f(Z) = e^{Zt}$ is an entire function and has the well known associated infinite power series expansion $p(Z)$

$$p(Z) = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \cdots$$

which satisfies all conditions of the above definition, it is permissible to define

$$[e^{At}] = I + t[A] + \frac{t^2}{2!}[A]^2 + \frac{t^3}{3!}[A]^3 + \ldots$$

where $[A]$ is any N x N matrix and t is the time variable (scalar), as the matrix exponential function.

Here, it is not intended to give a rigorous treatment of matrix functions but to present theorems and useful results that can be applied to discrete linear systems.

Definition: The minimal polynomial of the matrix $[A]$ is the polynomial $\varphi(\lambda)$ of least degree such that $\varphi(A) = 0$ and the coefficient of the highest power of λ is unity

$$\varphi(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{r_i} \qquad (2.1)$$

where λ_i, $i=1, 2, \ldots, k$ are the eigenvalues of $[A]$, r_i is the highest rank of any generalized eigenvector associated with the repeated eigenvalue λ_i, $i=1, 2, \ldots, k$.

From Eq. (2.1)

$$\frac{1}{\varphi(\lambda)} = \sum_{i=1}^{k} \frac{\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}}; \qquad \sum_{i=1}^{k} \eta_i(\lambda) \prod_{\substack{j=1 \\ \neq i}}^{k} (\lambda-\lambda_j)^{r_j} = 1 \tag{2.2}$$

where $\eta_i(\lambda)$ are polynomials in λ of degree less than r_i. Consider

$$\varphi_i(\lambda) = \frac{\varphi(\lambda)\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}} \tag{2.3}$$

From Eqs. (2.2) and (2.3) it can be shown that

$$\sum_{i=1}^{k} \varphi_i(\lambda) = 1 \tag{2.4}$$

$$\varphi_i(\lambda_i) = \eta_i(\lambda_i) \prod_{\substack{j=1 \\ \neq i}}^{k} (\lambda_i-\lambda_j)^{r_j} = 1 \tag{2.5}$$

$$i=1, 2, \ldots, k$$

$$\varphi_i^{\ell}(\lambda_i) = 0 \qquad \ell = 1, 2, \ldots (r_i-1) \tag{2.6}$$

$$\varphi_i^{\ell}(\lambda_j) = 0 \qquad \ell = 0, 1, \ldots r_j-1 \tag{2.7}$$

$$j = 1, 2, \ldots k, \neq i$$

$$i = 1, 2, \ldots k$$

Let $\left[E_i\right] = \varphi_i(\left[A\right])$

where $\varphi_i(\lambda)$ is defined by Eq. (2.3). $\left[A\right]$ and $\left[E_i\right]$ are N x N square matrices. ($k \leq N$)

It may be shown that

$$\left[E_i\right]\left[E_i\right] = \left[E_i\right] \qquad i = 1, 2, \ldots k \tag{2.8}$$

$$[E_\ell][E_i] = [0] \qquad \ell \neq i = 1, 2, \ldots k \tag{2.9}$$

$$\sum_{j=1}^{k} [E_i] = I \tag{2.10}$$

To demonstrate the validity of Eqs. (2.8), (2.9) and (2.10) use is made of the fact that in a matrix polynomial, the factors may be interchanged, e.g., to show Eq. (2.8). Consider

$$\varphi_i^2(\lambda) = \left(\frac{\varphi(\lambda)\,\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}}\right)^2$$

$$= \frac{\varphi^2(\lambda)\,\eta_i^{\,2}(\lambda)}{(\lambda-\lambda_i)^{2r_i}} - \frac{\varphi(\lambda)\,\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}} + \frac{\varphi(\lambda)\,\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}}$$

Let

$$g_i(\lambda) = \frac{\varphi^2(\lambda)\,\eta_i^{\,2}(\lambda)}{(\lambda-\lambda_i)^{2r_i}} - \frac{\varphi(\lambda)\,\eta_i(\lambda)}{(\lambda-\lambda_i)^{r_i}}$$

$$= \frac{\varphi^2(\lambda)\,\eta_i^{\,2}(\lambda) - \varphi(\lambda)\,\eta_i(\lambda)\,(\lambda-\lambda_i)^{r_i}}{(\lambda-\lambda_i)^{2r_i}}$$

$$= \frac{\prod_{\substack{j=1\\ \neq i}}^{k} (\lambda-\lambda_j)^{r_j}\,\eta_i(\lambda)(\lambda-\lambda_i)^{2r_i}\left(\prod_{\substack{j=1\\ \neq i}}^{k} (\lambda-\lambda_j)^{r_j}\,\eta_i(\lambda) - 1\right)}{(\lambda-\lambda_i)^{2r_i}}$$

But, from Eq. (2.2)

$$\left[\prod_{\substack{j=1 \\ \neq i}}^{k} (\lambda-\lambda_j)^{r_j} \eta_i(\lambda) - 1 \right] = - \sum_{\substack{\ell=1 \\ \neq i}}^{k} \eta_\ell(\lambda) \prod_{\substack{s=1 \\ \neq \ell}}^{k} (\lambda-\lambda_s)^{r_s}$$

$$\therefore\ g_i(\lambda) = -\eta_i(\lambda) \prod_{\substack{j=1 \\ \neq i}}^{k} (\lambda-\lambda_j)^{r_j} \sum_{\substack{\ell=1 \\ \neq i}}^{k} \eta_\ell(\lambda) \prod_{\substack{s=1 \\ \neq \ell}}^{k} (\lambda-\lambda_s)^{r_s} \qquad (2.11)$$

From Eq. (2.11) it is easy to see that $g_i([A]) = 0$ as each term of Eq. (2.11) contains a group of factors which when collected together are equal to $\varphi([A])$, i.e.,

$$g_i(\lambda) = \varphi(\lambda) \sum_{i=1}^{k} L_i(\lambda, \lambda_\ell), \quad \ell=1, 2, \ldots k$$

where $L_i(\lambda, \lambda_\ell)$ is a polynomial with factors $(\lambda-\lambda_\ell)$, $\ell=1, 2, \ldots k$.

As $\varphi([A]) = 0$

$$\left[\varphi_i([A])\right]^2 = [E_i][E_i] = \varphi_i([A]) = [E_i] \qquad (2.12)$$

Similiarly, Eqs. (2.9) and (2.10) can be verified.

Theorem: If f is analytic in an open set containing $\lambda_1 \lambda_2 \ldots \lambda_k$ then

$$f([A]) = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} \frac{1}{i!} \left[A-\lambda_\ell I\right]^i f^i(\lambda_\ell) [E_\ell] \qquad (2.13)$$

where $f([A])$ and E_ℓ are defined above.

Application of Matrix Functions to Discrete Systems

In 2N space the equations of motion of general linearly damped systems may be written as

$$I\dot{Z} + uZ = F(t) \tag{2.14}$$

where Z, u and F(t) are given by Eq. (1.117). To solve Eq. (2.14) using the matrix exponential function, first solve the homogeneous problem $F(t) = \{0\}$

$$I\dot{Z} + uZ = 0 \tag{2.15}$$

Let

$$\{Z\} = \left[e^{Bt}\right]\{Y\} \tag{2.16}$$

Substituting Eq. (2.16) into Eq. (2.15)

$$IB\left[e^{Bt}\right]Y + u\left[e^{Bt}\right]Y = \{0\}$$

$$\left[B + u\right]\left[e^{Bt}\right]Y = \{0\} \tag{2.17}$$

Note: It is easy to show from the definition of a matrix function that $\frac{d}{dt}\left[e^{Bt}\right] = \left[B\right]\left[e^{Bt}\right]$ and that $\left[e^{Bt}\right]$ is never singular.

From Eq. (2.17)

$$B = -u \tag{2.18}$$

for Eq. (2.16) to hold for arbitrary Y. Hence solution to Eq. (2.14) is

$$Z = \left[e^{-ut}\right]Y$$

$\{Y\}$ is obtained from the initial condition $\{Z\} = \{Z(0)\}$ at t=0. From the definition of $\left[e^{-ut}\right]$ it may be seen that $\left[e^{-ut}\right]_{t=0} = I$.

$$\therefore \{Z\} = \left[e^{-ut}\right]\{Z(0)\} \tag{2.19}$$

To solve the inhomogeneous problem of Eq. (2.14),

$$I\dot{Z} + uZ = F(t) \tag{2.20}$$

Consider

$$\{Z'\} = \int_0^t \left[e^{-u(t-\tau)}\right] \{F(\tau)\}\, d\tau \tag{2.21}$$

The kernel of the integral of Eq. (2.21) is a 2N x 1 column vector with elements which are functions of t and τ. The integral sign is interpreted as operating on each element of the matrix in turn.

$$\{\dot{Z}'\} = -\int_0^t u\left[e^{-u(t-\tau)}\right] F(\tau)\, d\tau + F(t)$$

Hence $\{Z'\}$ is the solution to the inhomogeneous problem with zero initial conditions. Therefore

$$\{Z\} = \left[e^{-ut}\right] Z(0) + \int_0^t \left[e^{-u(t-\tau)}\right] F(\tau) d\tau \tag{2.22}$$

is the complete solution to Eq. (2.14) when $\{Z(t)\}_{t=0} = \{Z(0)\}$.

Practical Computation of the Solution Using the Matrix Exponential Formulation

The infinite power series for $\left[e^{At}\right]$ is not a practical method of computation[25] with the possible exception of approximations for small t. The interpolation method, which is based directly on the definition of a matrix function is the first of three practical methods for the computation of $\left[e^{At}\right]$.

Let

$$\left[p(\left[A\right])\right] = \left[e^{At}\right] = \left[f(\left[A\right])\right] \tag{2.23}$$

where $p(\lambda)$ is a polynomial which satisfies the conditions given in the definition of the function of a matrix. The computation of $\left[e^{At}\right]$ involves the determination of an interpolating polynomial $p(\lambda)$ and the evaluation of $\left[p(A)\right]$. Since every matrix $\left[A\right]$ of order N has an associated polynomial $\varphi(\left[A\right])$ of degree $m \leq N$

$$\left[e^{At}\right] = \sum_{i=0}^{m-1} a_i(t) \left[A\right]^i \qquad m = \sum_{i=1}^{k} r_i \tag{2.24}$$

where $a_i(t)$ are functions of t, m is the degree of $\varphi(\left[A\right])$, k is the number of distinct eigenvalues of $\left[A\right]$ and r_i is the highest rank of any generalized eigenvector associated with λ_i - the i^{th} eigenvalue of A.

$$\therefore \quad p(\lambda) = \sum_{i=0}^{m-1} a_i \lambda^i$$

As $f(\lambda) = e^{\lambda t}$; $p(\lambda)$ must satisfy the following conditions

$$p^{\ell}(\lambda_j) = \sum_{i=\ell}^{m-1} \frac{i!}{(i-\ell)!} a_i \lambda_j^{i-\ell} = t^{\ell} e^{\lambda_j t} \tag{2.25}$$

$$\ell = 0, 1, \ldots r_{j-1}, \quad j = 1, 2, \ldots, k$$

As a simple example of the determination of e^{At}, let

$$\left[A\right] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.26}$$

As $\left[A\right]$ is in Jordan form it is easy to see that

$$\lambda_1 = 2 \qquad r_1 = 2$$

$$\lambda_2 = 1 \qquad r_2 = 1 \qquad m = 3$$

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$$

From Eq. (2.25)

$$\begin{aligned} &\ell=0 \quad j=1 \quad p(2) = a_0 + 2a_1 + 4a_2 = e^{2t} \\ &\ell=0 \quad j=2 \quad p(1) = a_0 + a_1 + a_2 = e^{t} \\ &\ell=1 \quad j=1 \quad p'(1) = \quad a_1 + 4a_2 = te^{2t} \end{aligned} \tag{2.27}$$

Solving Eq. (2.27) gives

$$a_0 = 4e^{t} - 3e^{2t} + 2te^{2t}, \quad a_1 = -4e^{t} + 4e^{2t} - 3te^{2t}$$

$$a_2 = e^{t} - e^{2t} + te^{2t}$$

But

$$A^2 = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Hence} \quad \left[e^{At}\right] = \begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

If the minimal polynomial has all distinct roots the interpolating polynomial is

$$p(\lambda) = \sum_{\ell=1}^{k} \frac{\prod_{\substack{i=1 \\ \neq \ell}}^{k} (\lambda - \lambda_i)}{\prod_{\substack{i=1 \\ \neq \ell}}^{k} (\lambda_\ell - \lambda_i)} f(\lambda_\ell) \tag{2.28}$$

i.e., the Lagrange interpolating polynomial.

Although Eq. (2.28) is suitable for digital computer computation the use of the interpolating formula when the minimal polynomial has repeated roots leads to equations of type Eq. (2.27) which are difficult to solve, for all t, on computers.

Computation Using the Fundamental Formula

From Eq. (2.13)

$$f([A]) = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} \frac{[A-\lambda_\ell I]^i}{i!} f^i(\lambda_\ell) E_\ell \tag{2.29}$$

Let

$$[R_{\ell i}] = \frac{1}{i!}[A-\lambda_\ell I]^i E_\ell$$

to simplify the notation. Hence Eq. (2.29) may be reduced to

$$f([A]) = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} f^i(\lambda_\ell)[R_{\ell i}] \tag{2.30}$$

From Eq. (2.30)

$$[e^{At}] = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} t^i e^{\lambda_\ell t}[R_{\ell i}] \tag{2.31}$$

Although the matrices $[R_{\ell i}]$ are expressible by the use of the quantities $\varphi_i(\lambda)$, defined above, as polynomials in $[A]$, the most efficient method of calculating $f([A])$ from Eq. (2.30) is by inserting appropriate trial functions f.

As an example consider

$$[A] = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 5 & 1 & 5 \end{bmatrix} \qquad \|[A-\lambda I]\| = (\lambda-2)^2(\lambda-1)$$
$$\varphi(\lambda) = (\lambda-2)^2(\lambda-1) \tag{2.32}$$

In actual fact the determinant of $[A-\lambda I]$ may be used instead of the minimal polynomial. If the degree of the minimal polynomial is less than N, the order of the matrix, the use of the determinant

results in some $[R_{\ell i}]$ being null matrices. In this case

$$\lambda_1 = 2, \quad r_1 = 2, \quad \lambda_2 = 1, \quad r_2 = 1 \quad k = 2$$

$$\therefore f([A]) = f(\lambda_1)[R_{10}] + f'(\lambda_1)[R_{11}] + f(\lambda_2)[R_{20}] \tag{2.33}$$

To determine $[R_{10}]$, $[R_{11}]$ and $[R_{20}]$

Consider

(i) $f(\lambda) = 1$ all λ $\quad \therefore f(\lambda_1) = f(\lambda_2) = 1, \; f'(\lambda_1) = 0$

$$\therefore I = [R_{10}] + [R_{20}] \tag{2.34}$$

(ii) $f(\lambda) = (\lambda - 2)$ $\quad \therefore f(\lambda_1) = 0, \; f(\lambda_2) = -1, \; f'(\lambda_1) = 1$

$$\therefore A - I = [R_{11}] - [R_{20}] \tag{2.35}$$

(iii) $f(\lambda) = (\lambda - 2)^2$ $\quad \therefore f(\lambda_1) = 0, \; f(\lambda_2) = 1, \; f'(\lambda_1) = 0$

$$\therefore [A - 2I]^2 = [R_{20}] \tag{2.36}$$

Solving for $[R_{10}]$, $[R_{11}]$ and $[R_{20}]$ from Eqs. (2.34), (2.35) and (2.36)

$$[R_{10}] = I - [A - 2I]^2 = 4A - A^2 - 3I$$

$$[R_{11}] = A - I - [4A - A^2 - 3I] = 2I + A^2 - 3A$$

$$[R_{20}] = [A - 2I]^2 = A^2 - 4A + 4I \tag{2.37}$$

From Eqs. (2.30), (2.31) and (2.37)

$$f([A]) = e^{At} = e^{\lambda_1 t}[R_{10}] + te^{\lambda_1 t}[R_{11}] + e^{\lambda_2 t}[R_{20}]$$

$$= e^{2t}\begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -4 & -2 & -2 \\ 2 & 1 & 1 \\ 6 & 3 & 3 \end{bmatrix} + e^{t}\begin{bmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \qquad (2.38)$$

It is easy to see that the trial functions, f=1, $(\lambda-2)$, $(\lambda-2)^2$ are obtained from the minimal polynomial $\varphi(\lambda)$, or $\|[A-\lambda I]\|$ in the case where $\varphi(\lambda)$ is not known, by successively cancelling one factor from $\varphi(\lambda)$, then another from the resulting quotient and so on until the final quotient is 1. It may be seen that with these trial functions the matrices $[R_{\ell i}]$ are readily evaluated, as the linear equations in $[R_{\ell i}]$ form a triangular set. Furthermore, the whole process is ideally suited for computer programming, but the details will not be developed here.

Computation Using Laplace Transform

A third method for the practical computation of the matrix exponential $[e^{At}]$ involves the use of Laplace transform methods.

Consider the following vector matrix equation

$$\{\dot{x}\} = [A]\{x\} \quad t \geq 0 ; \quad \{x(0)\} = I_0 \qquad (2.39)$$

where $\{x\}$ is a N x 1 column vector, with i^{th} element $x_i(t)$, and $[A]$ is a constant N x N matrix with ij element a_{ij}.

Taking the Laplace transform of Eq. (2.39)

$$s\{\bar{x}\} - x(0) = [A]\{\bar{x}\} \qquad (2.40)$$

where s is the Laplace transform variable and $\{\bar{x}\}$ is the Laplace transform of $\{x\}$, or

$$\left[sI-A\right]\left\{\bar{x}\right\} = I_o \tag{2.41}$$

$$\therefore \quad \left\{\bar{x}\right\} = \left[sI-A\right]^{-1} I_o \tag{2.42}$$

But the solution to Eq. (2.39) is $\left\{x\right\} = e^{At} I_o$. Hence the Laplace transform of $e^{At} = \left[sI-A\right]^{-1}$ (2.43)

It is possible to compute $\left[sI-A\right]^{-1}$ in the form

$$\left[sI-A\right]^{-1} = \frac{1}{\Delta(s)} \left[C(s)\right] \tag{2.44}$$

where $\Delta(s) = \left\|\left[sI-A\right]\right\|$ and C(s) is a N x N matrix with elements which are polynomials in s of degree (N-1) at most. Let

$$\frac{1}{\Delta(s)} \left[C(s)\right] = \left[R(s)\right] \tag{2.45}$$

where $\left[R(s)\right]$ is a matrix with elements which are rational functions of s.

It is possible to perform partial fraction expansions on the elements of $\left[R(s)\right]$, (as the denominator of each element is $\Delta(s)$) and to expand $\left[R(s)\right]$ in the form of a sum

$$\left[R(s)\right] = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} \left[B_{\ell i}\right] \frac{1}{(s-\lambda_\ell)^{i+1}} \tag{2.46}$$

where $\left[B_{\ell i}\right]$ are constant matrices.

Taking the inverse Laplace transform of Eq. (2.46)

$$\left[e^{At}\right] = \sum_{\ell=1}^{k} \sum_{i=0}^{r_\ell - 1} \left[B_{\ell i}\right] \frac{t^i}{i!} e^{\lambda_\ell t} \tag{2.47}$$

This method of computing $\left[e^{At}\right]$ depends on the evaluation of

the matrices $\left[B_{\ell i}\right]$. For large order systems the direct inversion of $\left[sI-A\right]$ and the use of the partial fraction expansion are not practical. However, there is an algorithm[25] available for determining $\left[C(s)\right]$ and $\left[B_{\ell i}\right]$ which may be used on a digital computer for large order systems.

Once $\left[e^{-ut}\right]$ is calculated, the solution to the general linear system , Eq. (2.22) , may be obtained in an obvious manner.

Stability of Linear Damped Systems

The question of the stability[8, 9, 10, 11, 32, 33] of linear systems is an interesting one, and has occupied the attention of engineers and mathematicians for many years. The problem is basically bound up with the algebraic sign of the real part of the roots of the frequency equation. The Routh-Hurwitz criteria[8, 9, 32, 33] on the coefficients of powers of λ in the frequency equation provide a very quick method for determining if a system is stable. In the feedback control field, use is made of the Nyquist Plot[8, 32] and the Root Locus[8, 32] plot for stability analysis. Both of these techniques are suitable for the determination of the stability of the systems under discussion in this work. The Nyquist plot, essentially the mapping of the transfer function along the Nyquist contour, is readily drawn and an analysis of the stability of the system is easily made. The method is based on a theorem in complex variable due to Cauchy. The Root Locus, as its name implies, is a plot of the roots as some parameter of the system is varied. The control engineer has available techniques, which

allow him to graph the root locus with the minimum of calculation.

Besides the actual stability of the system, the degree of stability (i. e., a measure of the magnitude of the negative real part of the roots) is of interest to the design engineer. It is for this reason that engineers in general, prefer either the Nyquist Plot or the Root Locus Plot to the Routh-Hurwitz method for the analysis of the stability of systems. By its very nature the Root Locus Plot indicates the degree of stability as a function of the parameter varied. By changing the Nyquist contour it is possible to determine the degree of stability from a Nyquist Plot. However, the Routh-Hurwitz criteria indicates if the system is stable but does not give any measure of the degree of stability.

Mathematicians have been interested in the problem of the inertia of matrices[34, 36] for some years. The inertia of a matrix is a three-tuple quantity (a, b, c), where a is the number of eigenvalues of the matrix with positive real parts, b is the number of purely imaginary eigenvalues and c is the number of eigenvalues with negative real parts. There are many theorems available on the conservation of the inertia of matrices under various transformations and on sets of matrices which have the same inertia characteristics. Unfortunately, these results are too general for the specific problem of the stability of a given matrix -- the question normally of interest to the dynamicist. However, these theorems on the inertia of matrices are invaluable to the linear economic model analyst,[35] and it is to be expected that within a few years some new results, on the inertia of matrices, of

interest to workers in mechanics, will be available.

At the present time, the most powerful analytical tool for studying the stability of systems is undoubtedly a technique based on the work of Liapunov.[9, 10, 11] In fact, he proved the two theorems that will be used in the analysis of the stability of the systems under discussion in this work.

Necessary and Sufficient Conditions for the Stability of Linear Damped Systems

In N space the equations of motion of linear damped systems may be written as

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{2.48}$$

where M, C and K are N x N matrices, and x is a N x 1 column vector.

From Eq. (2.48) the frequency equation may be obtained by substituting $x = e^{\lambda t}\{\varphi\}$

$$e^{\lambda t}\left[\lambda^2 M + \lambda C + K\right]\{\varphi\} = 0 \tag{2.49}$$

For non trivial solutions of Eq. (2.49)

$$\left\|\left[\lambda^2 M + \lambda C + K\right]\right\| = 0 \tag{2.50}$$

Equation (2.50) is the frequency equation in N space.

In 2N space, provided M^{-1} exists, the equations of motion may be written as

$$I_{2N}\dot{Z} + \begin{bmatrix} M^{-1}C & M^{-1}K \\ -I_N & 0 \end{bmatrix} Z = 0 \tag{2.51}$$

where $\{Z\} = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}$ a 2N x 1 column vector, and I_N is the identity matrix

of order N. By substituting $Z = e^{\lambda t}\{\Phi\}$ in Eq. (2.51), the frequency equation in 2N space may be determined

$$\left\| \lambda I_{2N} + \begin{bmatrix} M^{-1}C & M^{-1}K \\ -I_N & 0 \end{bmatrix} \right\| = 0 \qquad (2.52)$$

or

$$\left\| \begin{bmatrix} M^{-1}C + \lambda I & M^{-1}K \\ -I_N & \lambda I_N \end{bmatrix} \right\| = 0 \qquad (2.53)$$

Multiplying each of the first N columns of the determinant in Eq. (2.53) by λ and adding the (N+i)th column to the i^{th} (i=1, 2, . . . , N)

$$\frac{1}{(\lambda)^N} \left\| \begin{bmatrix} \lambda^2 I_N + \lambda M^{-1}C + M^{-1}K & M^{-1}K \\ 0 & \lambda I_N \end{bmatrix} \right\| = 0 \qquad (2.54)$$

Hence, the frequency equation in 2N space reduces to

$$\left\| \left[\lambda^2 I_N + \lambda M^{-1}C + M^{-1}K \right] \right\| = 0$$

which is easily seen to be the frequency equation in N space (Eq. (2.50), provided M^{-1} exists.

From the above work, on the analysis of systems in 2N space, it is shown that if

$$\left[R\right] = \begin{bmatrix} M^{-1}C & M^{-1}K \\ -I & 0 \end{bmatrix}, \quad \text{a 2N x 2N matrix} \qquad (2.55)$$

is diagonalizable, the motion of the system is a summation of solutions of type

$$\begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}_i = A_i e^{-d_i t} \begin{Bmatrix} -d_i \{\phi_i\} \\ \{\phi_i\} \end{Bmatrix} \tag{2.56}$$

Hence, if and only if the real part of d_i, the i^{th} diagonal element of the reduced diagonal form of $[R]$, is greater than or equal to zero is the i^{th} mode stable. The elements along the diagonal of the reduced form of $[R]$ are the eigenvalues of $[R]$ -- therefore on utilizing the concept of superposition of solutions of type (2.56) it may be shown that if $[R]$ is reducible to a strictly diagonal matrix it is both necessary and sufficient for stability of the system that the real parts of the eigenvalues of $[R]$ be greater than or equal to zero. As the eigenvalues of $[R]$ are in fact the negative of the roots of the frequency equation this condition may be stated: If $[R]$ is reducible to diagonal form, it is both necessary and sufficient that the roots of the frequency equation have non positive real parts for the system to be stable.

The situation is somewhat different when the reduced form of $[R]$ is not strictly diagonal as in this case there is coupling between the reduced equations. If the reduced form of $[R]$ is not strictly diagonal then there are at least 2 equations in the reduced systems of equations having the form

$$\begin{aligned} \dot{\eta}_i + d_i \eta_i &= 0 \\ \dot{\eta}_{i-1} + d_i \eta_{i-1} + \eta_i &= 0 \end{aligned} \tag{2.57}$$

for some i $\quad 1 < i \leq N$. From Eq. (2.57)

$$\eta_i = A_i e^{-d_i t}$$

$$\eta_{i-1} = B_{i-1} e^{-d_i t} - A_i t e^{-d_i t} \qquad (2.58)$$

Hence, unless the real part of d_i is strictly positive the mode associated with η_{i-1} is unstable, due to the t factor in the second term of Eq. (2.58). Therefore using the principle of superposition it is easy to show that if the reduced form of $[R]$ is not strictly diagonal it is necessary and sufficient that the real parts of the eigenvalues associated with the generalized eigenvectors of $[R]$ be strictly positive for the system to be stable. This fact points out one of the difficulties of using any of the standard methods for determining the marginal stability of the system. The marginal stability curve or the stability boundary is normally determined by calculating the values of the parameters of the system which make the real part of the least stable root of the frequency equation zero. The system may or may not be stable on the stability boundary as normally defined. It is entirely a matter of the eigenvectors (ordinary or generalized) associated with the root whose real part is zero.

If the definition of stability is extended to the forced vibration problem, i.e., a system is stable if to a bounded input the output is bounded, it is easy to see that the conditions derived above are also necessary and sufficient conditions for the forced vibration problem to be stable. For if the system is diagonalizable in 2N space, the contribution of the forcing function to the motion of the system is a summation of terms of type

$$\begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}_i = \left(\int_0^t e^{-d_i(t-\tau)} g_i(\tau) d\tau \right) \begin{Bmatrix} -d_i \{\phi_i\} \\ \{\phi_i\} \end{Bmatrix} \tag{2.59}$$

As $g_i(\tau)$ is the forcing function of the i^{th} uncoupled equation, it is a finite sum of bounded quantities and therefore is itself bounded. Hence, it is possible to bound the left hand side of Eq. (2.59) as follows

$$\max_{[0,t]} \left| \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}_i \right| \leq \max_{[0 \leq \tau \leq t]} \left| g_i(\tau) \right| \int_0^t e^{-d_i(t-\tau)} d\tau \left| \begin{Bmatrix} -d_i \{\phi_i\} \\ \{\phi_i\} \end{Bmatrix} \right| \tag{2.60}$$

$$\leq \max_{[0 \leq \tau \leq t]} \left| g_i(\tau) \right| \frac{1}{d_i} (1 - e^{-d_i t}) \left| \begin{Bmatrix} -d_i \{\phi_i\} \\ \{\phi_i\} \end{Bmatrix} \right| \tag{2.61}$$

where $\left| \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}_i \right|$ indicates the vector with elements equal to the moduli of the corresponding elements of $\begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}$. Therefore, unless $d_i = 0$ it is necessary and sufficient that the real part of d_i (the i^{th} eigenvalue of $[R]$) be non negative for the left hand side to be bounded for all bounded $g_i(\tau)$. The case of $d_i = 0$ corresponds mathematically to a zero eigenvalue and physically to rigid body motion. Needless to remark, rigid body motion falls under the classification of instability when one uses the above definition of stability (e. g., the rigid body motion $x = at$ is unstable as x increases without bound for a bounded input). By the use of the superposition properties of linear systems it may be shown that a linear lumped parameter system, which is diagonalizable in 2N space, is stable if and only if the eigenvalues of the system have non negative real parts.

Turning now to the question of necessary and sufficient conditions for the stability of the forced vibration problem when the reduced form of $[R]$ is not strictly diagonal it will be shown that it is possible to develop analogous results to those derived above. From Eq. (1.137) it may be seen that

$$\begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}_i = \left(\int_0^t e^{-d_i \tau} g_i(t-\tau)\, d\tau + \int_0^t (t-\tau) e^{-d_i (t-\tau)} g_{i+1}(\tau)\, d\tau \right) \{\Phi\}_i \tag{2.62}$$

is the contribution of the forcing function to the motion of the system in the i^{th} mode if the i+1 eigenvector is a generalized eigenvector. Proceeding in a manner similar to that used above it is possible to show that the left hand side of Eq. (2.62) is bounded when $g_i(t)$ and $g_{i+1}(t)$ are bounded (all t) if and only if the real part of d_i is strictly negative. Using the concept of superposition it may be shown that linear lumped parameter systems, which are not reducible to strictly diagonal form in 2N space, are stable if and only if the eigenvalues of the system associated with the generalized eigenvectors have strictly positive real parts.

Use of Liapunov's Stability Theorems on Linear Lumped Parameter Systems

Before using Liapunov's work to develop sufficient conditions for stability of linear lumped parameter systems the following two theorems are stated:

Theorem I: Let $[Y]$, a N x N matrix, be determined by the relation

$$[A]^T[Y] + [Y][A] = -I \tag{2.63}$$

Then a necessary and sufficient condition that the real N x N matrix $[A]$ be a stability matrix is that $[Y]$ be positive definite. A stability matrix is one whose eigenvalues have negative real parts. From the above formulation of the equations of motion of the linear damped problem in 2N-space it is seen that the matrix

$$-[R] = \begin{bmatrix} -M^{-1}K & -M^{-1}C \\ I & 0 \end{bmatrix} = [u] \quad \text{a } 2N \text{ x } 2N \text{ matrix} \tag{2.64}$$

must be a stability matrix for the homogeneous solution to decay asymptotically to zero as $t \to \infty$. It was also shown that for a bounded forcing function, the response is bounded for all time if $[u]$ is a stability matrix. If one of the eigenvalues has zero real part, the mode associated with this eigenvalue in the homogeneous response of the system does not decay as $t \to \infty$ but it is always bounded provided there are no generalized eigenvectors associated with this eigenvalue.

To determine the stability of a system by means of Theorem I, it is necessary to solve Eq (2.63) and to test if the matrix $[Y]$ is positive definite. For large order systems this is a formidable task and in general a digital computer is needed to facilitate the calculations. The solution to Eq. (2.63) is

$$\left[Y\right] = \int_0^\infty e^{At}\, e^{A^T t}\, dt \qquad (2.65)$$

provided $\lambda_i + \lambda_j \neq 0$ $i, j = 1, 2, \ldots N$ where λ_i are the eigenvalues of $\left[A\right]$. In the calculation of the integral in Eq. (2.65) recourse may be had to the methods described above for computing e^{At}. To test if the matrix $\left[Y\right]$ is positive definite is a routine task and can be done by either determining the eigenvalues of $\left[Y\right]$ or the signs of the principal minors of $\left[Y\right]$.

Although Theorem I is extremely useful as a practical test for the stability of systems it is of little value as an analytical tool for determining the classes of matrices which are stable. However, Liapunov also proved another theorem (Theorem II) which is stated below, and which can be used for developing sufficient conditions for matrices to be stable. Theorem II forms the basis of what is generally known as Liapunov's Direct Method for the determination of stability. Before stating the theorem it is necessary to define:

Liapunov Function (V(x)): A function V(x) of the elements of a vector $x = (x_1, x_2, \ldots x_N)$ is said to be a Liapunov function if

(i) V(x) is continuous together with its first and second partial derivatives in an open region S about the origin.

(ii) $V(0) = 0$.

(iii) At any point of S, except the origin, V(x) is strictly positive.

(iv) For x in S, $\dot{V}(x) \leq 0$.

Conditions (i), (ii), and (iii) are necessary and sufficient for V(x) to be positive definite. Generally, Liapunov functions are constructed from consideration of the physical system one is studying. Kinetic and potential energy terms, being positive definite, play a large role in the development of Liapunov functions. However, considerable ingenuity and familiarity is required to utilize Theorem II to full advantage, particularly when dealing with non-linear systems.

Theorem II: If there exists in some neighborhood S of the origin a Liapunov function V(x), then the origin is stable. If moreover $-\dot{V}(x)$ is positive definite in S the stability is asymptotic.

Whether the stability is asymptotic or not depends on whether the eigenvalues are complex or purely imaginary. In either case, Theorem II can only give sufficient conditions for the stability of the system. Although the theorem is not constructive, a considerable amount of work has been done on the stability of systems using the results of the theorem and thereby experience has been gained in the development of appropriate Liapunov functions for particular types of systems. As stated, the theorem gives only sufficient conditions for stability and it is therefore entirely possible that the sufficient conditions for stability derived by the use of one Liapunov function be more restrictive than the sufficient conditions derived by the application of a different Liapunov function.

Application of Theorem II to Derive Sufficient Conditions for the Stability of Linear Damped Systems

Consider the equations of motion of a multi-degree of freedom linear system in N space

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{2.66}$$

Equation (2.66) may be written in 2N space with Z a 2N x 1 vector as the variable, but for present purposes the N space form of Eq. (2.66) is more suitable.

In Eq. (2.66) let M and K be both N x N symmetric matrices and C a general real matrix. Consider the function

$$V(x,\dot{x}) = \dot{x}^T M\dot{x} + \left\{x^T C^T + \dot{x}^T M\right\}\left\{Cx + M\dot{x}\right\} + x^T Kx \tag{2.67}$$

As M and K are symmetric matrices, the first and third terms of the right hand side of Eq. (2.67) are twice the usual kinetic and potential energies, respectively, of the system. The second term of the right hand side of Eq. (2.67) is the inner product of a vector with itself and is therefore always non negative. $V(x, \dot{x})$ is a Liapunov function if

(i) M and K are symmetric and positive definite.

or (ii) M is symmetric and positive definite

K is symmetric and non-negative definite

C is non-singular

x, $\dot{x}$ being vectors with real elements.

That $V(x, \dot{x})$ is a positive definite function under condition (i) follows from the fact that the first and third terms on the right hand of Eq. (2.67) are zero if and only if $x = \dot{x} = 0$ and the second term is

always non-negative. By examining the second term on the right hand side of Eq. (2.67)

$$D = \left\{x^T C^T + \dot{x}^T C\right\}\left\{Cx + M\dot{x}\right\}$$
$$= \left\{x^T C^T Cx\right\} \quad \text{if } \dot{x} = 0 \tag{2.68}$$

It may be shown that $V(x, \dot{x})$ as defined by Eq. (2.67) is a positive definite function under condition (ii). For, if M is symmetric and positive definite and K is symmetric and non-negative definite, $V(x, \dot{x})$ is positive for $\dot{x} \neq 0$ all x, as in this case the first term on the right hand side is positive while the second and third terms are always non-negative. If $\dot{x} = 0$ and $x \neq 0$ the first term is zero and the third term can be zero, but, from Eq. (2.68), the second term cannot be zero if C is non-singular. To show that the system is stable, consider

$$\dot{V}(x, \dot{x}) = \ddot{x}^T M\dot{x} + \dot{x}^T M\ddot{x} + \left\{\dot{x}^T C^T + \ddot{x}^T M\right\}\left\{Cx + M\dot{x}\right\}$$
$$+ \left\{x^T C^T + \dot{x}^T M\right\}\left\{C\dot{x} + M\ddot{x}\right\} + \dot{x}^T Kx + x^T K\dot{x} \tag{2.69}$$

Now from Eq. (2.66)

$$M\ddot{x} = -\left\{C\dot{x} + Kx\right\}$$
$$\ddot{x}^T M = -(\dot{x}^T C^T + x^T K) \tag{2.70}$$

On substituting Eq. (2.70) into (2.69)

$$\dot{V}(x, \dot{x}) = -\dot{x}^T C^T\dot{x} - \dot{x}^T C\dot{x} - x^T KCx - x^T C^T Kx - x^T KM\dot{x} - \dot{x}^T MKx \tag{2.71}$$

But

$$x^T KM\dot{x} = \dot{x}^T MKx$$

$$\therefore \quad \dot{V}(x, \dot{x}) = -2x^T MKx - \dot{x}^T\left[C + C^T\right]\dot{x} - x^T\left[KC + C^T K\right]x \tag{2.72}$$

But from (i) and (ii) M and K are symmetric and M is positive definite, therefore a transformation Q exists such that

$$Q^T Q = I, \quad Q^T M Q = \bar{M}, \qquad \text{a diagonal matrix with all positive diagonal elements}$$

$$Q^T K Q = \bar{K}, \qquad \text{a diagonal matrix with all non-negative diagonal elements.}$$

$$M = Q\bar{M}Q^T; \qquad K = Q\bar{K}Q^T$$

$$\therefore \; x^T MKx = x^T Q\bar{M}Q^T Q\bar{K}Q^T x = Y^T \bar{M}\bar{K}Y$$

where

$$Y = Q^T x.$$

Due to the form of $\bar{M}$ and $\bar{K}$, the matrix $\bar{M}\bar{K}$ is a diagonal matrix with all non-negative diagonal terms

$$\therefore \; Y^T \bar{M}\bar{K} Y \geq 0 \qquad \text{all } Y, \text{ i.e., all } x \text{ (as } x = QY)$$

Hence $\dot{V}(x, \dot{x}) \leq 0$ if

$$\left[C + C^T\right] \quad \text{and} \quad \left[KC + C^T K\right] \tag{2.73}$$

are both non-negative definite matrices. Thus the linear lumped parameter system described by Eq. (2.66) is stable in Liapunov's sense if the specifications on the system parameters given by (i) or (ii) and (2.73) are satisfied. That these conditions are sufficient though not necessary for the system to be stable may be seen by the following example:

Let the specifications on the system parameters be:

(iii) M and K be symmetric positive definite matrices, and C be a symmetric non-negative definite matrix. These specifications, (iii), include specifications (i) above. The remaining condition, (2.73), for stability requires that

$$\left[KC + CK\right] \quad \text{be a non-negative definite matrix} \tag{2.74}$$

But it is easy to show that no further conditions are required in this case (Sylvester first proved this) for the system to be stable. For, taking

$$V(x,\dot{x}) = \dot{x}^T M\dot{x} + x^T Kx \tag{2.75}$$

$V(x, \dot{x})$ as defined by Eq. (2.75) is a positive definite function as M and K are positive definite matrices.

$\dot{V}(x,\dot{x}) = -2\dot{x}^T C\,\dot{x} \leq 0$ as C is non-negative definite

Hence $V(x, \dot{x})$ as defined by Eq. (2.75) is a proper Liapunov function and the system is therefore stable. However, the conditions (iii) specified on M, C and K do not guarantee the non-negative definiteness of Eq. (2.74). A numerical example clearly shows this

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \tag{2.76}$$

$$\therefore \quad CK + KC = \begin{bmatrix} 2 & -11 \\ -11 & 20 \end{bmatrix}$$

which is not non-negative definite and hence condition (2.75) is not satisfied in this case. Moreover, it is easy to show by using Eq. (2.75) that if M and K are both symmetric and positive definite matrices the system described by Eq. (2.66) is stable if $\left[C + C^T\right]$ is a non-negative definite matrix.

As a final example of the use of Liapunov's method the following theorem is proved.

Theorem: Let the canonical form of the equation of motion of a linear lumped parameter system be

$$I\ddot{x} + C\dot{x} + Kx = 0 \tag{2.77}$$

where

$$C = C_s + C_{ss};\quad C_s = \frac{1}{2}\left[C + C^T\right]$$

$$K = K_s + K_{ss};\quad C_{ss} = \frac{1}{2}\left[C - C^T\right]$$

subscript s indicates the symmetric part, subscript ss indicates the skew-symmetric part of the matrix. Then if K_s is positive definite and

$$\left[\begin{array}{c|c} C_s & K_{ss} \\ \hline K_{ss}^T & \left[K^T C\right]_s \end{array}\right]$$

is non-negative definite, stability is assured. This, as before, is a sufficient, though not a necessary, condition.

Proof: Define the function

$$V(x, \dot{x}) = \dot{x}^T\dot{x} + \left\{x^T C^T + \dot{x}^T\right\}\left\{Cx + \dot{x}\right\} + x^T\left[K + K^T\right]x \tag{2.78}$$

As the first and second terms of Eq. (2.78) are the inner products of vectors with themselves and $K + K^T = 2K_s$ is a positive definite matrix, $V(x, \dot{x})$ is a positive definite function.

$$\dot{V}(x, \dot{x}) = \ddot{x}^T\dot{x} + \dot{x}^T\ddot{x} + \left\{\dot{x}^T C^T + \ddot{x}^T\right\}\left\{Cx + \dot{x}\right\} + \left\{x^T C^T + \dot{x}^T\right\}\left\{C\dot{x} + \ddot{x}\right\}$$
$$+ \dot{x}^T\left[K + K^T\right]x + x^T\left[K + K^T\right]\dot{x} \tag{2.79}$$

But from Eq. (2.77)

$$\ddot{x} = -\left\{C\dot{x} + Kx\right\}\ ;\quad \ddot{x}^T = -\left(\dot{x}^T C^T + x^T K^T\right) \tag{2.80}$$

$$\dot{x}^T C^T + \ddot{x}^T = -x^T K^T \qquad \text{and } C\dot{x} + \ddot{x} = -Kx$$

On substituting Eq. (2.80) into Eq. (2.79)

$$V(x, \dot{x}) = -(\dot{x}^T \left[C+C^T\right]\dot{x} + x^T \left[-K+ K^T\right]\dot{x} + \dot{x}^T \left[K-K^T\right]x$$
$$+ x^T \left[K^T C + C^T K\right] x)$$
$$= -2(\dot{x}^T C_s \dot{x} + x^T K_{ss} \dot{x} + \dot{x} K_{ss}^T x + x^T \left[K^T C\right]_s \dot{x})$$

$$\therefore\ V(x, \dot{x}) = -2 \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}^T \left[\begin{array}{c|c} C_s & K_{ss} \\ \hline K_{ss}^T & \left[K^T C\right]_s \end{array}\right] \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} \qquad (2.81)$$

Hence if

$$\left[\begin{array}{c|c} C_s & K_{ss} \\ \hline K_{ss}^T & \left[K^T C\right]_s \end{array}\right] \quad \text{is a non-negative definite matrix} \qquad (2.82)$$

$V(x, \dot{x})$ is a Liapunov function of the system described by Eq. (2.77) under the given restriction that K_s be positive definite. This completes the proof of the theorem quoted above.

It is well to note that condition (2.82) implies that C_s be positive definite. If K is a symmetric matrix $K_{ss} = 0$, and condition (2.82) reduces to

$$\left[\begin{array}{c|c} C_s & 0 \\ \hline 0 & \left[KC\right]_s \end{array}\right] \quad \text{is a non-negative definite matrix}$$

i.e., $\left[C + C^T\right]$ and $\left[KC + C^T K\right]$ be non-negative definite matrices -- sufficient conditions already derived under the same specifications above.

Sufficient Conditions for Instability of the System

Liapunov proved the following theorem on the instability of systems.

Theorem III: If $V(x, \dot{x})$, with $V(0, 0) = 0$, has continuous first partial derivatives in S, a region enclosing the origin, then the origin is unstable if $\dot{V}(x, \dot{x})$ is positive definite in S and $V(x, \dot{x})$ is able to assume positive values arbitrarily near the origin. Like the two previous theorems on stability, it is possible to use this theorem to derive sufficient conditions for the instability of systems.

Consider the following theorem:

Theorem: For the canonical system

$$I\ddot{x} + C\dot{x} + Kx = 0 \tag{2.83}$$

If C is a symmetric positive definite matrix, and the symmetric part of K is a negative definite matrix the system is unstable.

Proof: Define the (non-energy) function

$$V(x, \dot{x}) = \dot{x}^T x + \frac{1}{2} x^T C x \tag{2.84}$$

$V(0, 0) = 0$ and $V(x, \dot{x})$ has continuous first partial derivatives in the entire x-$\dot{x}$ hyperplane.

$V(x, \dot{x})$ is positive provided $\dot{x}^T x$ is positive

$$\dot{V}(x, \dot{x}) = \ddot{x}^T x + \dot{x}^T \dot{x} + \frac{1}{2} \dot{x}^T C x + \frac{1}{2} x^T C \dot{x}$$

But from Eq. (2.83)

$$\ddot{x}^T = -(\dot{x}^T C^T + x^T K^T)$$

$$\therefore\ V(x,\dot{x}) = -x^{T}\frac{(K^{T}+K)}{2}x + \dot{x}^{T}\dot{x} = -x^{T}K_{s}x + \dot{x}^{T}\dot{x}$$

Hence if K_s is a negative definite matrix

$$\dot{V}(x,\dot{x}) \text{ is positive all } x, \dot{x} \neq 0$$

Therefore as $V(x,\dot{x})$, defined by Eq. (2.84), satisfies all conditions of Theorem III, the system described by Eq. (2.83) under the given restrictions is unstable in the part of the x-$\dot{x}$ hyperplane where $x^{T}\dot{x}$ is positive.

Further Results on Stability Conditions

When the lumped parameter linearly damped system is formulated in 2N space

$$\frac{dZ}{dt} = AZ \tag{2.85}$$

the matrix A has no obvious characteristics (e.g., it is not symmetric, all its elements have not the same algebraic sign, etc.) except that it may be partitioned into N x N matrices involving combinations of the original matrices in the N space formulation. The application of Theorem I (Liapunov's first stability theorem) to the matrix A of Eq. (2.85) does not produce any easily checked criterion that would provide necessary and sufficient conditions for stability. In fact even if A has some special form, (e.g., it is symmetric) there are very few readily available necessary and sufficient conditions that may be applied to test for stability. With this in mind, the following results of previous investigators[37] have been gathered in the form of statements, to show what is available.

Definitions:

A is said to be Hickian if all the principal minors of order r has the algebraic sign of $(-1)^r$.

A is said to be quasi-negative definite if $x^T Ax$ is negative for all real non null vectors x.

$$\text{As } x^T Ax = x^T A^T x$$

A will be quasi-negative if and only if the symmetric matrix $A + A^T$ is negative definite.

A is said to have a quasi-dominant main diagonal if there exists a vector $\{h\}$ with all positive elements h_i, $i=1, 2, \ldots 2N$ such that

$$|a_{jj}|\, h_j > \sum_{\substack{i=1 \\ \neq j}}^{2N} |a_{ij}|\, h_i \qquad \text{or} \qquad |a_{ii}|\, h_i > \sum_{\substack{j=1 \\ \neq i}}^{2N} |a_{ij}|\, h_j$$

Each a_{ii} is negative $\quad i, j=1, 2, \ldots, 2N$

A is said to be Metzlerian if a_{ii} is negative and a_{ij} is positive $i \neq j$ $i, j=1, 2, \ldots, 2N$.

The following statements, of interest to dynamicists, have already been proved.

(1) Equation (2. 85) being stable does not require that A be Hickian.

(2) A being Hickian does not imply that Eq. (2. 85) is stable.

(3) A being quasi-negative definite implies that Eq. (2. 85)is stable.

(4) A being quasi-negative definite implies that A is Hickian.

(5) A being Hickian does not imply that A is quasi-negative definite.

(6) If A is symmetric, it is necessary and sufficient for the stability of Eq. (2. 85) that A be Hickian (i. e., negative definite).

(7) If A is Metzlerian, it is necessary and sufficient for the stability of Eq. (2. 85) that A be Hickian.

(8) Equation (2. 85) being stable does not imply that A is quasi-negative definite.

(9) A being Metzlerian does not imply that A is quasi-negative definite.

(10) A having a dominant main diagonal (h_i=1 all i) does not imply that A is quasi-negative definite.

(11) A having a quasi-dominant main diagonal does not imply that A is quasi-negative definite.

(12) If A is Metzlerian, then a necessary and sufficient condition for stability of Eq. (2. 85) is that A have a quasi-dominant main diagonal.

(13) Equation (2. 85) is stable if A has a dominant main diagonal.

These results, may be applied to the systems of interest in this work, provided A has the appropriate form. However, A rarely has a suitable form and it is in this context that the sufficient conditions for stability derived above have proved very useful.

Perturbation Theory

This section is intended to show the power of matrix perturbation techniques when applied to linear damped systems. The motivation for the use of these techniques is that quite often the analyst is required to estimate the effect, on the response of the system, of a

small change in one or more of the matrices describing the system. Here, two examples of the use of the technique are given. The unperturbed system in each case is restricted to be composed of symmetric matrices. This restriction is not due to the breakdown of the technique when applied to more general systems, but is an attempt to present the main ideas with a minimum of algebraic complexities. For the application of perturbation analysis to the general linear system reference should be made to a report by Dr. Caughey[16] and the present author.

Perturbation Theory Applied to Undamped Systems

Consider the equations of motion of an undamped multi-degree of freedom vibrating system

$$M\ddot{x} + Kx + \epsilon K^{*}x = 0 \tag{2.86}$$

where M and K are symmetric and positive definite N x N matrices, ϵ is a small quantity, and K^{*} is any N x N matrix. Assume a solution of the form

$$x = \bar{\phi}^{n} e^{\bar{\lambda}_n t} \tag{2.87}$$

Substitute Eq. (2.87) into Eq. (2.86)

$$\left[\bar{\lambda}_n^2 M + K + \epsilon K^{*}\right] \bar{\phi}^{n} = 0 \tag{2.88}$$

Assume that it is possible to expand $\bar{\phi}^{n}$ and $\bar{\lambda}_n$ into a Taylor series expansion in ϵ about $\epsilon = 0$

$$\begin{aligned} \bar{\lambda}_n &= \lambda_n + \epsilon \mu_n + \epsilon^2 \upsilon_n + \ldots . \\ \bar{\phi}^{n} &= \phi^{n} + \epsilon \psi^{n} + \epsilon^2 \Omega^{n} + \ldots . \end{aligned} \tag{2.89}$$

It is well to remark that expansions of the type given by Eq. (2. 89) are not always valid as the implicit assumption of such an expansion is that a small change in a matrix produces only a small change in the eigenvectors and eigenvalues. Whereas this assumption is valid in the case of the eigenvalues, it is not necessarily so in the case of the eigenvectors; for consider

$$K = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \; ; \qquad K^* = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$\left\| \left[K - \lambda I \right] \right\| = (\lambda - 5)^2$, i. e., the eigenvalues of K are 5, 5, K has one ordinary eigenvector ϕ^1 and one generalized eigenvector ϕ^2

$$\phi^1 = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix} \; ; \qquad \phi^2 = \frac{1}{5} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \tag{2.90}$$

$\left\| \left[K + \varepsilon^2 K^* - \lambda I \right] \right\| = (\lambda - 5 - 2\varepsilon)(\lambda - 5 + 2\varepsilon)$, i. e., the eigenvalues of $\left[K + \varepsilon^2 K^* \right]$ are $\lambda = 5 + 2\varepsilon$, $5 - 2\varepsilon$, the eigenvectors

$$\bar{\phi}^1 = \begin{Bmatrix} \varepsilon - 1 \\ 2 \end{Bmatrix} \; ; \qquad \bar{\phi}^2 = \begin{Bmatrix} -1 - \varepsilon \\ 2 \end{Bmatrix} \tag{2.91}$$

From Eqs. (2. 90) and (2. 91) it is easy to see that an expansion of type (2. 89) is valid for $\bar{\phi}^1$ but not for $\bar{\phi}^2$. In general it may be shown that the series given by (2. 89) is valid for small ε provided ϕ^n and $\bar{\phi}^n$ are not associated with repeated eigenvalues. In cases where ϕ^n is associated with a repeated eigenvalue but the matrix has a full complement of ordinary eigenvectors reference should be made to the work cited above by Dr. Caughey[16] and the present

author. The main difficulty in these cases is to determine which of the infinite set of eigenvectors to select as the zeroth order solution in the perturbed system.

Substituting Eq. (2.89) into Eq. (2.86)

$$\left[(\lambda_n^2 + 2\epsilon\lambda_n\mu_n + \epsilon^2(\mu_n^2 + 2\lambda_n\nu_n))\, M + K + \epsilon K^*\right]\left\{\phi^n + \epsilon\psi^n + \epsilon^2\Omega^n + \ldots\right\} = 0 \tag{2.92}$$

Collecting powers of ϵ and equating them separately to zero:

$$\epsilon^0 : \left[\lambda_n^2 M + K\right]\phi^n = 0 \tag{2.93}$$

$$\epsilon^1 : \left[\lambda_n^2 M + K\right]\psi^n = -2\mu_n\nu_n M\phi^n - K^*\phi^n \tag{2.94}$$

$$\epsilon^2 : \left[\lambda_n^2 M + K\right]\Omega^n = -2\lambda_n\nu_n M\phi^n - K^*\psi^n \tag{2.95}$$

The zeroth order solution, Eq. (2.93), is merely the unperturbed solution. Provided λ_n is a simple eigenvalue, there is no difficulty as in this case ϕ^n has a unique direction. This assumption will be made here.

The first order perturbation solution may be obtained as follows: Expanding

$$\psi^n = \sum_{i=1}^{N} a_{nj}\phi^j \qquad n=1, 2, \ldots, N \tag{2.96}$$

Equation (2.96) is a valid expansion because the eigenvectors of the zeroth order system, $\left[\lambda_n^2 M + K\right]\phi^n$ form a basis for the space as M and K are symmetric and positive definite. Due to these properties of the zeroth order system the following conditions may be deduced

$$\phi^{n^T} M \phi^k = 0 \qquad n \neq k$$

$$\phi^{n^T} M \phi^n = 1 \qquad n, k=1, 2, \ldots, N \tag{2.97}$$

$$\phi^{n^T} K \phi^k = 0$$

$$\phi^{n^T} K \phi^n = \lambda_n^2$$

$$\lambda_n^2 \geq 0 \qquad n=1, 2, \ldots, N \tag{2.98}$$

ϕ^n are essentially real vectors

Premultiply Eq. (2.94) by ϕ^{ℓ^T}

$$\lambda_n^2 \phi^{\ell^T} M \psi^n + \phi^{\ell^T} K \psi^n = -2\mu_n \lambda_n \phi^{\ell^T} M \phi^n - \phi^{\ell^T} K^* \phi^n \tag{2.99}$$

Transpose Eq. (2.93) (noting that M and K are symmetric) and post multiply by ψ^n

$$\lambda_\ell^2 \phi^{\ell^T} M \psi^n + \phi^{\ell^T} K \psi^n = 0 \tag{2.100}$$

Subtract Eq. (2.100) from Eq. (2.99)

$$(\lambda_n^2 - \lambda_\ell^2) \phi^{\ell^T} M \psi^n = -2\mu_n \lambda_n \delta_{n\ell} - \phi^{\ell^T} K^* \phi^n \tag{2.101}$$

where

$$\delta_{n\ell} = 1 \qquad n = \ell$$

$$= 0 \qquad n \neq \ell \qquad n, \ell=1, 2, \ldots, N$$

From Eq. (2.101) if $n=\ell$

$$\mu_n = \frac{-\phi^{n^T} K^* \phi^n}{2\lambda_n} \tag{2.102}$$

$n \neq \ell$. on substituting for ψ^n from Eq. (2.96)

$$a_{n\ell} = - \frac{\phi^{\ell T} K^* \phi^n}{(\lambda_n^2 - \lambda_\ell^2)} \qquad (2.103)$$

$$n, \ell = 1, 2, \ldots, N \quad n \neq \ell$$

To determine a_{nn}, $n=1, 2, \ldots, N$, recourse must be had to an appropriate normalization criterion, e. g.,

$$\phi^{nT} M \phi^n = 1 \qquad (2.104)$$

i. e.,

$$\left\{\phi^{nT} + \varepsilon\psi^{nT} + \varepsilon^2 \Omega^{nT} \ldots.\right\} [M] \left\{\phi^n + \varepsilon\psi^n + \varepsilon^2 \Omega^n + \ldots.\right\} = 1$$

$$\phi^{nT} M \phi^n + 2\varepsilon\psi^{nT} M \phi^n + \varepsilon^2 (\quad) + \ldots. \quad = 1$$

But from Eq. (2.97)

$$\phi^{nT} M \phi^n = 1$$

$$\therefore \psi^{nT} M \phi^n = 0$$

But from Eq. (2.96)

$$\psi^{nT} M \phi^n = a_{nn} \qquad n=1, 2, \ldots, N$$

$$\therefore a_{nn} = 0 \qquad n=1, 2, \ldots, N \qquad (2.105)$$

Substituting Eqs. (2.102), (2.103) and (2.105) into Eq. (2.89)

$$\bar{\lambda}_n = \lambda_n - \varepsilon \frac{\phi^{nT} K^* \phi^n}{2\lambda_n} + \varepsilon^2 (\quad) + \ldots.$$

$$\bar{\phi}^n = \phi^n - \varepsilon \sum_{\substack{j=1 \\ \neq n}}^{N} \frac{\phi^{j^T} K^* \phi^n}{\lambda_n^2 - \lambda_j^2} \phi^j + \varepsilon^2 (\quad) + \ldots . \qquad n=1, 2, \ldots, N \tag{2.106}$$

From the above analysis, it is clear that the following restrictions on the zeroth order system are required

(i) the eigenvalues be distinct

(ii) μ_n the first order correction be small, i.e., $\lambda_n = 0(1)$

(iii) $a_{n\ell}$ $n, \ell = 1, 2, \ldots, N$ be finite, i.e., $\lambda_n^2 - \lambda_\ell^2 = 0(1)$.

The case in which the eigenvalues are not distinct has been discussed above. The problems associated with restrictions (ii) and (iii) cannot be completely eliminated, although some work has been done in this area.[38, 39]

As the eigenvalues are imaginary and K^* is a real matrix

$$\lambda_n = \sqrt{-1}\,\omega_n \qquad n=1, 2, \ldots, N$$

From Eq. (2.106)

$$\bar{\lambda}_n = \sqrt{-1}\left(\omega_n + \varepsilon \frac{\phi^{n^T} K^* \phi^n}{2\omega_n} + \varepsilon^2 (\quad) \ldots .\right) \tag{2.107}$$

$$\therefore \text{ if } \quad \phi^{n^T} K^* \phi^n > 0 \qquad |\bar{\lambda}_n| > \omega_n$$

$$\phi^{n^T} K^* \phi^n < 0 \qquad |\bar{\lambda}_n| < \omega_n \tag{2.108}$$

Hence the frequency of oscillation of the n^{th} mode, $\bar{\lambda}_n$, may be greater or less than the corresponding frequency of the unperturbed system.

Perturbation Analysis of Systems with Damping

The equations of motion of a linear damped system may be written as

$$M\ddot{x} + \varepsilon C\dot{x} + Kx = 0 \tag{2.109}$$

Assuming for algebraic convenience that the system described by Eq. (2.109) is passive, i.e., M, K and C are symmetric matrices and M is positive definite. As shown in Chapter I, there are two mutually exclusive possibilities -- either the system described by Eq. (2.109) is classically or non-classically damped. If the system is classically damped the following analysis holds. Let

$$x = \bar{\Phi}\xi \tag{2.110}$$

where

$$\bar{\Phi}^T M \bar{\Phi} = I, \quad \bar{\Phi}^T K \bar{\Phi} = \bar{K}, \quad \text{a diagonal matrix}$$

and

$$\bar{\Phi}^T C \bar{\Phi} = \bar{C}, \quad \text{a diagonal matrix.}$$

By the usual methods, on substituting Eq. (2.110) into Eq. (2.109) and pre-multiplying by $\bar{\Phi}$, the system is uncoupled.

$$I\ddot{\xi} + \varepsilon \bar{C}\dot{\xi} + \bar{K}\xi = 0 \tag{2.111}$$

i.e., the i^{th} equation of Eq. (2.111) is

$$\ddot{\xi}_i + \varepsilon \bar{C}_i \dot{\xi}_i + \omega_i^2 \xi_i = 0 \qquad i=1, 2, \ldots, N$$

where $\bar{C}_i$ and ω_i^2 are the i^{th} diagonal elements of $\bar{C}$ and $\bar{K}$, respectively

$$\therefore \quad \bar{\lambda}_i = -\frac{\varepsilon}{2}\,\bar{C}_i \pm \sqrt{\omega_i^2 - \left(\frac{\varepsilon \bar{C}_i}{2}\right)^2}$$

i.e.,

$$\omega_{iD} = \omega_{iu}\sqrt{1 - \left(\frac{\varepsilon \bar{C}_i}{2\omega_{iu}}\right)^2} \leq \omega_i \tag{2.112}$$

where ω_{iD} and ω_{iu} are the damped ($\varepsilon \neq 0$) and the undamped ($\varepsilon = 0$) natural frequency of the i^{th} mode of the system respectively. From Eq. (2.112) it may be seen that for classical systems the natural frequency of any mode of the damped system is at most equal to the natural frequency of the corresponding mode in the undamped system. If the system specified by Eq. (2.109) is not classically damped the following perturbation analysis approximates very closely the exact solution under rather general conditions on the matrices of the system. It is, moreover, considerably less tedious than the exact solution obtainable by the methods described in Chapter I. Assume

$$x = e^{\bar{\lambda}_n t}\,\bar{\Phi}^n$$

with

$$\bar{\Phi}^n = \phi^n + \varepsilon\psi^n + \varepsilon^2\Omega^n + \ldots.$$

and

$$\bar{\lambda}_n = \lambda_n + \varepsilon\mu_n + \varepsilon^2\nu_n + \ldots. \tag{2.113}$$

Substituting Eq. (2.113) in (2.109)

$$\left[\bar{\lambda}_n^2 M + \varepsilon\bar{\lambda}_n + K\right]\bar{\Phi}^n = 0 \tag{2.114}$$

Expanding

$$\bar{\lambda}_n^2 = \lambda_n^2 + \varepsilon(2\lambda_n\mu_n) + \varepsilon^2(2\lambda_n\upsilon_n+\mu_n^2) + \ldots. \tag{2.115}$$

By Eqs. (2.113), (2.114) and (2.115)

$$(\lambda_n^2+\varepsilon(2\lambda_n\mu_n) + \varepsilon^2(2\lambda_n\upsilon_n+\mu_n^2) + \ldots.)\, M\{\phi^n+\varepsilon\psi^n+\varepsilon^2\Omega^n+\ldots.\}$$

$$+\varepsilon(\lambda_n+\varepsilon\mu_n+\varepsilon^2\upsilon_n+\ldots.)\, C\{\phi^n+\varepsilon\psi^n+\varepsilon^2\Omega^n+\ldots.\}$$

$$+K\{\phi^n+\varepsilon\psi^n+\varepsilon^2\Omega^n+\ldots.\} = 0 \tag{2.116}$$

Collecting powers of ε and equating each independently to zero:

$$\varepsilon^0: \left[\lambda_n^2M+K\right]\phi^n = 0 \tag{2.117}$$

$$\varepsilon^1: \left[\lambda_n^2M+K\right]\psi^n = -(2\lambda_n\mu_n\phi^n+\lambda_nC\phi^n) \tag{2.118}$$

$$\varepsilon^2: \left[\lambda_n^2M+K\right]\Omega^n = -(2\lambda_n\upsilon_n+\mu_n^2)M\phi^n-2\lambda_n\mu_nM\psi^n$$

$$-\lambda_nC\psi^n-\mu_nC\phi^n \tag{2.119}$$

From Eq. (2.117) it may be seen that ϕ^n and λ_n are the eigenvectors and eigenvalues, respectively, of the undamped system specified by Eq. (2.109). It may, therefore be concluded that

$$\lambda_n^2 < 0 \qquad \text{all } n=1,2,\ldots,N$$

ϕ^n are essentially real vectors

$$\begin{aligned}
&\phi^{nT}K\phi^{\ell} = 0 && n\neq\ell\\
&\phi^{nT}M\phi^{\ell} = 0 && n\neq\ell\\
&\phi^{nT}M\phi^{n} = 1
\end{aligned} \tag{2.120}$$

It is well to note that, as before, for algebraic simplicity, the eigenvalues of the undamped problem are assumed to be distinct. To determine μ_n, n=1, 2, . . . , N, pre-multiply Eq. (2.118) by $\phi^{\ell T}$

$$\phi^{\ell T}\left\{\lambda_n^2 M\psi^n + K\psi^n\right\} = -2\lambda_n\mu_n\phi^{\ell T}M\phi^n - \lambda_n\phi^{\ell T}C\phi^n \tag{2.121}$$

But from Eq. (2.117)

$$\phi^{\ell T}\left\{\lambda_\ell^2 M\psi^n + K\psi^n\right\} = 0 \tag{2.122}$$

On subtracting Eq. (2.122) from Eq. (2.121)

$$(\lambda_n^2 - \lambda_\ell^2)\,\phi^{\ell T}M\psi^n = -2\lambda_n\mu_n\delta_{\ell n} - \lambda_n\phi^{\ell T}C\phi^n \tag{2.123}$$

On setting $n=\ell$

$$\mu_n = -\frac{1}{2}\left(\phi^{nT}C\,\phi^n\right) \qquad n=1, 2, \ldots, N \tag{2.124}$$

On setting $n\neq\ell$ in Eq. (2.123)

$$\phi^{\ell T}M\psi^n = -\frac{\phi^{\ell T}C\phi^n}{\lambda_n^2 - \lambda_\ell^2}\,\lambda_n \qquad n, \ell = 1, 2, \ldots, N \tag{2.125}$$

Expanding ψ^n, n=1, 2, . . . , N in terms of the ϕ^j, j=1, 2, . . . , N, the eigenvectors of the undamped system (they form a complete set)

$$\psi^n = \sum_{j=1}^{N} a_{nj}\phi^j \tag{2.126}$$

On substituting Eqs. (2.126) and (2.120) into Eq. (2.125)

$$a_{n\ell} = \phi^{\ell T} M \psi^n = -\lambda_n \frac{\phi^{\ell T} C \phi^n}{\lambda_n^2 - \lambda_\ell^2} \qquad n \neq \ell \tag{2.127}$$

The coefficients a_{nn}, $n=1, 2, \ldots, N$ may be determined by the use of the normalized condition

$$\bar{\phi}^{n T} M \bar{\phi}^n = 1$$

from which, as in the case of the first example of the use of perturbation analysis, $a_{nn}=0$, $n=1, 2, \ldots, N$.

In a similar manner the second order perturbation parameters may be obtained. Letting

$$Q^n = \sum_{j=1}^{N} b_{nj} \phi^j \tag{2.128}$$

it may be shown that

$$b_{n\ell} = \frac{1}{\lambda_n^2 - \lambda_\ell^2} \left(\frac{1}{2} (\phi^{n T} C \phi^n)(\phi^{\ell T} C \phi^n) + \sum_{\substack{j=1 \\ \neq n}}^{N} \frac{\lambda_n^2}{\lambda_n^2 - \lambda_j^2} (\phi^{j T} C \phi^n)(\phi^{\ell T} C \phi^j) \right)$$

$$n \neq \ell$$

$$b_{nn} = \phi^{n T} M Q^n = -\frac{1}{2} \sum_{j=1}^{N} a_{nj}^2$$

$$v_n = \frac{1}{8\lambda_n} (\phi^{n T} C \phi^n)^2 + \frac{1}{2} \sum_{\substack{j=1 \\ \neq n}}^{N} \frac{\lambda_n}{\lambda_n^2 - \lambda_j^2} (\phi^{j T} C \phi^n)^2 \tag{2.129}$$

$$n, \ell = 1, 2, \ldots, N$$

From Eqs. (2.113), (2.126) and (2.128), the eigenvectors of the damped system, to terms of order ε^2, are

$$\bar{\phi}^n = \phi^n + \varepsilon \sum_{j=1}^{N} a_{nj}\phi^j + \varepsilon^2 \sum_{i=1}^{N} b_{nj}\phi^j \qquad (2.130)$$

where a_{nj}, b_{nj}, $n, j=1, 2, \ldots, N$ are given by Eqs. (2.127) and (2.129). If the system is classically damped

$$\phi^{n^T} C \phi^{\ell} = 0 \qquad n \neq \ell \qquad n, \ell = 1, 2, \ldots, N$$

and therefore in this case $a_{n\ell} = b_{n\ell} = 0$, $n \neq \ell$, i. e.,

$$\bar{\phi}^n = \phi^n \qquad n=1, 2, \ldots, N \qquad (2.131)$$

a well known result for classical systems.

Equation (2.131) merely states that the formulae derived above for non-classical systems are correct in the limit as the non-classically damped systems become classical.

If the system is non-classically damped, then

$$\phi^{n^T} C \phi^{\ell} \neq 0 \qquad n \neq \ell \qquad (2.132)$$

in general. Noting the properties of λ_n and ϕ^n, $n=1, 2, \ldots, N$ as given by Eq. (2.120), and remembering that the matrix C has real elements it is easy to show (from Eq. (2.127) and Eq. (2.129)) that

$a_{n\ell}$ are purely imaginary numbers

$b_{n\ell}$ are real numbers $\qquad n, \ell = 1, 2, \ldots, N$

Hence, from Eq. (2.131), the eigenvectors of the perturbed system may be written as

$$\bar{\phi}^n = \underbrace{\phi^n + \varepsilon^2 (\text{Real Vector})}_{\text{Real Vector}} + \underbrace{\sqrt{-1}\,(\varepsilon(\text{Real Vector}))}_{\text{Imaginary Vector}} + \varepsilon^3(\ldots) \qquad (2.133)$$

So that the first order correction to ϕ^n is imaginary and the second order correction is real.

Likewise, considering the eigenvalues of the perturbed systems:

$$\bar{\lambda}^n = \lambda_n + \varepsilon\mu_n + \varepsilon^2\upsilon_n$$

$$= \lambda_n - \varepsilon\left(\frac{1}{2}\phi^{n^T} C \phi^n\right) + \varepsilon^2\left(\frac{1}{8\lambda_n}\left(\phi^{n^T} C \phi^n\right)^2 + \frac{1}{2}\sum_{\substack{j=1 \\ \neq n}}^{N} \frac{\lambda_n}{\lambda_n^2 - \lambda_j^2}\left(\phi^{j^T} C \phi^n\right)^2\right)$$

$$+ \ldots.$$

But remembering that λ_n is purely imaginary:

$$\lambda_n = \sqrt{-1}\,\omega_n$$

$$\bar{\lambda}_n = \left(-\frac{1}{2}\varepsilon\phi^{n^T} C \phi^n + O(\varepsilon^3)\right) + \sqrt{-1}\left(\omega_n - \varepsilon^2\left(\frac{1}{8\omega_n}\left(\phi^{n^T} C \phi^n\right)^2 + \frac{1}{2}\sum_{\substack{j=1 \\ \neq n}}^{N} \frac{\left(\phi^{j^T} C \phi^n\right)^2 \omega_n}{\omega_n^2 - \omega_j^2}\right) + O(\varepsilon^4)\right) + \ldots. \qquad (2.135)$$

From Eq. (2.135) it is seen that the correction to the frequency of the perturbed system is of $O(\varepsilon^2)$.

Ordering the natural frequencies of the unperturbed system as follows

$$\omega_1 < \omega_2 < \omega_3 \ldots. < \omega_N \quad \text{(all distinct)}$$

It can be seen from Eq. (2.135)

$$\omega_{ND} = \omega_{Nu}\left(1 - \varepsilon^2\left(\frac{1}{8\omega_{Nu}^2}\left(\phi^{N^T} C \phi^N\right)^2 + \frac{1}{2}\sum_{j=1}^{N-1} \frac{\phi^{j^T} C \phi^N}{\omega_N^2 - \omega_j^2}\right)\right) + O(\varepsilon^4)$$

$$\leq \omega_{Nu} \qquad (2.136)$$

Likewise

$$\omega_{1D} = \omega_{1u}\left(1 + \frac{\varepsilon^2}{2}\sum_{j=2}^{N}\frac{(\phi^j C\phi^1)^2}{\omega_j^2 - \omega_1^2} - \frac{\varepsilon^2}{8\omega_{1u}^2}(\phi^{1^T} C\phi^1)^2\right) \tag{2.137}$$

Hence from Eq. (2.137) if

$$4\sum_{j=2}^{N}\frac{\omega_1^2}{\omega_j^2 - \omega_1^2}(\phi^{j^T} C\phi^1)^2 \geq (\phi^{1^T} C\phi^1)^2 \tag{2.138}$$

$$\omega_{1D} \geq \omega_{1u}$$

To sum up, the damped natural frequency of the highest mode is less than or equal to the undamped frequency and the damped natural frequency of the lowest mode may be higher than the undamped frequency depending on the form of the damping matrix and the mode separation. For the above analysis to be valid it is sufficient that

(i) $\varepsilon < 1$

(ii) the eigenvalues of both the unperturbed and the perturbed systems be distinct and of O(1)

(iii) the mode separation be such that $\lambda_j^2 - \lambda_i^2 = 0(1) \quad i \neq j$

$$i, j = 1, \ldots, N$$

Perturbation analysis may be used to advantage if a system has been solved exactly and the effect of small changes in any of the parameters of the system is to be determined. If the original system has repeated eigenvalues or small mode separation the algebraic complexity of the perturbation analysis does not warrant

its use over the direct solution of the perturbed problem except in cases where an analysis of the system over a range of variation in ε (e. g., $0 < \varepsilon < 1$) is envisioned. It should be noted that any solved system may be taken as the zeroth order system and that it is possible to perturb any or all of the matrices simultaneously.

Bounds for Eigenvalues

A problem of some interest in the present work is the rapid determination of close bounds for the eigenvalues of a matrix. A considerable amount of work[22, 23, 24, 26, 27] has been done in this area, dealing particularly with matrices possessing certain properties (e. g., normal matrices). Unfortunately, very little can be said about bounds for the eigenvalues of general matrices. A few well known results will first be quoted, followed by some developments and adaptations of these results to the linear damped problem. Finally, recently proved theorems of matrix analysis will be discussed and applied to linear systems. The advantage of having techniques available for the determination of close bounds on the eigenvalues of a matrix is that it would be possible to determine approximately the damping and natural frequency of the modes of a linear system without actually solving the problem analytically. It would also be possible to determine the stability of the system and the effect of changes in the parameters of the system.

Given the eigenvalue problem

$$Ax = \lambda Bx \tag{2.139}$$

where A and B are both N x N real symmetric matrices and B is positive definite,

$$R(u) = \frac{u^T A u}{u^T B u} \tag{2.140}$$

is known as Rayleigh's Quotient; u is any N x 1 real vector. The eigenvalues, λ_i, of Eq. (2.139) are all positive and are ranked as follows

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \qquad (\text{all} \geq 0)$$

The following results are well known.

(i) $R(u) \geq \lambda_1$, u any N x 1 vector

(ii) $R(u_i) \geq \lambda_i$ where u_i is orthogonal in B to the eigenvectors corresponding to $\lambda_1, \lambda_2, \cdots \lambda_{i-1}$

(iii) $R(u) \leq \lambda_N$.

These results may be used to determine the range of the natural frequencies of passive linear undamped systems. Consider the system

$$M\ddot{x} + Kx = 0$$

where M and K are N x N symmetric positive definite matrices. Let

$$x = e^{\sqrt{-1}\omega t}\phi$$

$$\therefore \; -\omega^2 M\phi + K\phi = 0 \tag{2.141}$$

$$\omega^2 M\phi = K\phi$$

Direct application of the above results to Eq. (2.141) yield

$$\omega_1^2 \leq \frac{u^T K u}{u^T M u} \leq \omega_N^2, \qquad u \text{ any N x 1 vector}$$

where ω_1 and ω_N are the lowest and highest natural frequencies, respectively, of the system. Therefore by maximizing and minimizing

the Rayleigh quotient for the system, it is possible to determine the range of the natural frequencies of the N normal modes. This maximizing and minimizing process may be done by trial and error, by approximating the eigenvectors associated with the lowest and highest natural frequency (using physical insight and past experience) or by some formal mathematical procedure such as the Ritz Method. The Ritz Method merely determines the best possible set of scalar multiplicitive factors for a given set of base vectors.

An extension of these results may be applied to the linear damped passive system

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{2.142}$$

where M, C and K are symmetric matrices and M is positive definite. Let

$$x = e^{a_i t}\phi^i \tag{2.143}$$

$$\therefore \quad a_i^2 M\phi^i + a_i C\phi^i + K\phi^i = 0 \tag{2.144}$$

Equation (2.144) is satisfied provided a_i and ϕ^i are eigenvalues and corresponding ordinary eigenvectors, respectively, of the system. If a_i is a repeated eigenvalue and ϕ^i is a generalized eigenvector associated with a_i, Eq. (2.144) is not satisfied. Premultiply Eq. (2.144) by $\bar{\phi}^{iT}$ the vector whose elements are the complex conjugates of the corresponding elements of ϕ^{iT}

$$a_i^2 \bar{\phi}^{iT} M\phi^i + a_i \bar{\phi}^{iT} C\phi^i + \bar{\phi}^{iT} K\phi^i = 0 \tag{2.145}$$

As M, C, and K are real matrices, this quadratic equation in a_i has real coefficients. If a_i, the i^{th} eigenvalue is complex, the solution to

Eq. (2.144) is a_i and $\bar{a}_i$

$$\therefore \left|a_i\right|^2 = \frac{\bar{\phi}^{i\,T} K \phi^i}{\bar{\phi}^{i\,T} M \phi^i} \quad ; \quad -2\,R\ell\, a_i = \frac{\bar{\phi}^{i\,T} C \phi^i}{\bar{\phi}^{i\,T} M \phi^i} \tag{2.146}$$

now $\{\phi^i\} = \{a_i\} + \sqrt{-1}\{b_i\}$ where $\{a_i\}$ and $\{b_i\}$ are N x 1 real vectors

$$\therefore \left|a_i\right|^2 = \frac{\{a_i - \sqrt{-1} b_i\}^T K \{a_i + \sqrt{-1} b_i\}}{\{a_i - \sqrt{-1} b_i\}^T M \{a_i + \sqrt{-1} b_i\}}$$

Therefore applying the above results

$$\omega_1^2 \leq \left|a_i\right|^2 \leq \omega_N^2 \qquad i=1, 2, \ldots, N \tag{2.147}$$

where ω_1 and ω_N are the lowest and highest natural frequencies of the system specified by

$$M\ddot{x} + Kx = 0$$

Similarly, it may be shown that

$$\lambda_1 \leq -2\,R\ell\, a_i \leq \lambda_N \qquad i=1, 2, \ldots, N \tag{2.148}$$

or

$$-\frac{\lambda_N}{2} \leq R\ell\, a_i \leq -\frac{\lambda_1}{2}$$

where λ_1 and λ_N are the lowest and highest frequencies associated with the system

$$M\ddot{y} + Cy = 0$$

From Eqs. (2.147) and (2.148) it is possible to determine bounds for the complex eigenvalues (both damping and frequency) of the passive system specified by Eq. (2.142). The annulus between circles of radius ω_1 and ω_N specifies the magnitude of a_i considered as a

radius vector, while the two lines perpendicular to the real axis at distances of $-\lambda_1/2$ and $-\lambda_N/2$ depicts the range of damping in the normal modes of the system. It is interesting to note that, as λ_1 is never negative for passive systems, Eq. (2.148) shows that such systems are always stable -- a result already shown. Further, it may be seen from Eq. (2.147) that the natural frequency of the damped lowest mode may be greater than the natural frequency of the corresponding mode of the undamped problem but that the natural frequency of the damped highest mode must be at most equal to the natural frequency of the corresponding mode of the undamped system. These observations check with the results of the perturbation analysis presented above.

The sum of the diagonal elements of a matrix, called the trace of the matrix, is equal to the sum of the eigenvalues of the matrix. Hence, if the system is classical the following method may be used to determine the average damping and average frequency of the normal modes.

Premultiply Eq. (2.142) by M^{-1}

$$I\ddot{x} + M^{-1}C\dot{x} + M^{-1}Kx = 0 \tag{2.149}$$

The average damping is given by

$$\frac{1}{2N}\sum_{i=1}^{N}\bar{C}_i = \frac{1}{2N}\ (\text{trace of } M^{-1}C) \tag{2.150}$$

The average of the magnitudes of the complex frequencies is given by

$$\frac{1}{N}\sum_{i=1}^{N}\left|a_i\right|^2 = \frac{1}{N}\ \text{(trace of } M^{-1}K) \tag{2.151}$$

Likewise from the 2N formulation of non-classical systems it may be seen that

$$\frac{1}{2N}\sum_{i=1}^{2N} a_i = \frac{1}{2N}\quad \text{(trace of } M^{-1}C) \tag{2.152}$$

where a_i is the i^{th} eigenvalue in the 2N formulation. If the eigenvalues are complex Eq. (2.152) reduces to the same form as the N-space classical equation for the average damping (Eq. (2.150)).

Other Theorems on Eigenvalue Bounds

The Courant-Fischer min-max theory -- an extension of the Rayleigh Quotient Analysis used above -- has many practical applications. One of the difficulties of using the Rayleigh Quotient to estimate the eigenvalues of a matrix is that some of the eigenvectors must be determined if a close upper bound on the intermediate eigenvalues is desired. The Courant-Fischer min-max theorem on the eigenvalues $\lambda_1 \leq \lambda_2 \leq , \ldots \leq \lambda_N$ of semi-definite systems given by Eq. (2.139), states that

$$\lambda_{N-r+1} = \left(\min_{q_s}\left(\max_{u}\ \frac{u^T A u}{u^T B u}\right)\right) \tag{2.153}$$

where u must satisfy the N-r conditions of orthogonality

$$q_s^T u = 0 \qquad s=1, 2, \ldots, N-r$$

and the vectors q_s are arbitrary.

It may be seen from the statement of this theorem that the eigenvectors are not needed to obtain close bounds on the intermediate eigenvalues. If the system is classical or only slightly non-classical (i.e., the damping matrix $C = C_1 + \varepsilon C_2$ where C_1 is classical) this theorem may be used to determine approximate values for the damping and moduli of the complex frequencies of each mode of the system. The values determined will be upper bounds in the case of classical systems. If, for example, it is desired to estimate the damping in the (N-r+1)-th mode:

Consider

$$\min_{q_s} \left(\max_{u} \frac{u^T C u}{u^T M u} \right) \tag{2.154}$$

where u must satisfy the N-r conditions

$$q_s{}^T u = 0$$

q_s, s=1, 2, . . . , N-r are any arbitrary set of vectors. From the Courant-Fischer theorem, Eq. (2.154) gives an upper bound for the (N-r+1)-th eigenvalue of the system

$$\lambda M x = C x$$

for any given set of q_s, s=1, 2, . . . , N-r. Hence if the system is classical, Eq. (2.154) can be used to give an upper bound on the damping in the (N-r+1)-th mode and if only slightly non-classical it may be used to approximate the damping in the (N-r+1)-th mode. An analogous expression may be derived for the magnitude of the complex frequency of any mode. In passing, it is well to note that no matter

what kind of system is being analyzed (e. g., classical, non-classical, reducible to strictly non-diagonal Jordan form, etc.) the trace of $M^{-1}C$ is equal to the sum of the complex frequencies and the product of the eigenvalues of $M^{-1}K$ is equal to the product of the complex frequencies.

There are a number of inclusion[26] and separation[24] theorems on the eigenvalues of symmetric matrices. The inclusion theorems are useful in depicting intervals along the real axis in which the eigenvalues of a matrix may fall. Separation theorems relate the eigenvalues of a matrix to the eigenvalues of lower order matrices obtained by the elimination of rows and columns from the original matrix. The 2N formulation of the linear damped problem gives, in general, a non definite self adjoint eigenvalue problem, i.e.,

$$\lambda_1 \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} Z_1 = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} Z_1$$

Let

$$R = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \quad ; \qquad S = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} \tag{2.155}$$

If it is assumed that M, C and K are symmetric, R and S are both symmetric but neither of them is positive definite. As the theorems on the bounds of eigenvalues of systems of type (2.155) require that either R or S be positive definite (strictly positive) it is reasonable to expect that rather poor bounds can only be obtained for the complex frequencies of non-classically damped systems. However, there are

some results[27] available for general matrices and these will be given. First it is necessary to premultiply Eq. (2.155) by R^{-1} to obtain the standard simple eigenvalue-eigenvector equation

$$\lambda_1 Z_1 = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix} Z_1 \qquad (2.156)$$

Let

$$\left[U_{ij}\right] = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix} \qquad \text{a } 2N \times 2N \text{ matrix}$$

Let

$$a = \max_{ij} \left|U_{ij}\right|$$

The crudest estimate for the bound of the absolute value of any eigenvalue of Eq. (2.156) is

$$\left|\lambda\right| \leq 2Na \qquad (2.157)$$

By separating U into a symmetric matrix U_s and a skew symmetric matrix U_{ss}, where

$$\left[U_{ij}\right] = \left[U_{sij}\right] + \left[U_{ssij}\right]$$

and let

$$b = \max_{ij} U_{sij} \; ; \qquad C = \max_{ij} U_{ssij}$$

$$\lambda_i = \gamma_i + \sqrt{-1}\,\beta_i \qquad i=1, 2, \ldots, 2N,$$

it is possible to derive better bounds for the eigenvalues

$$\left|\gamma_i\right| \leq 2Nb; \qquad \beta_i \leq 2NC \qquad \text{all } i \qquad (2.158)$$

Estimates may be further improved by using Frobenius' Theorem

$$|\lambda_i| \leq \max_{\ell}\left(\sum_{j=1}^{2N} |U_{\ell j}|\right); \quad |\lambda_i| \leq \max_{j}\left(\sum_{\ell=1}^{2N} |U_{\ell j}|\right) \quad \text{all } i$$

$$|\lambda_i| \geq \min_{\ell}\left(|a_{\ell\ell}| - \sum_{\substack{j=1 \\ \neq \ell}}^{2N} |a_{\ell j}|\right); \quad |\lambda_i| \geq \min_{j}\left(|a_{jj}| - \sum_{\substack{\ell=1 \\ \neq j}}^{2N} |a_{\ell j}|\right)$$

Bendixon's Theorem yields bounds closer than Eq. (2.158)

$$m \leq \gamma_i < M$$

$$|\beta_i| \leq C\sqrt{N(2N-1)} \tag{2.159}$$

where m and M are the smallest and largest eigenvalues, respectively, of U_s.

Pick's Theorem gives slightly better bounds for $|\beta_i|$

$$\begin{aligned} &\text{(i)} \quad r < \beta_i < R \\ &\text{(ii)} \quad |\beta_i| \leq C \ \operatorname{ctg} \pi/4N \quad \text{all } i \end{aligned} \tag{2.160}$$

where r and R are the smallest and largest eigenvalues of $-i\,U_{ss}$. It is possible to show that Eq. (2.160) (ii) are best possible general bounds for the imaginary part of the eigenvalues.

Further bounds[24, 26] may be obtained by the use of the concept of the field of values. The field of values is merely the set of values of the Rayleigh Quotient:

$$R(x) = \frac{x^T U x}{x^T x}$$

as x is varied, subject to an obvious modification if U is a matrix

with complex elements. However, as the theorems on the field of values usually apply to normal matrices, it is necessary to associate with U the matrix

$$A = \left[a_{ij}\right] = U^T U$$

A is symmetric and positive definite. The following inequalities hold:

$$a_1 \leq \left|\lambda_i\right|^2 \leq a_{2N} \tag{2.161}$$

$$\left|\lambda_i\right| \leq \left\{\sum_{i=1}^{2N} \sum_{j=1}^{2N} \left|a_{ij}\right|^2\right\}^{1/2}$$

where a_1 and a_{2N} are the lowest and highest eigenvalues, respectively, of A.

The results quoted in this section show the type of bounds that are available for the eigenvalues of systems of interest in this work. In general these bounds are too wide and they suffer from the serious defect of being bounds for the entire set of eigenvalues rather than for one particular eigenvalue. However, Eq. (2.154) is an attempt to isolate one particular eigenvalue and determine bounds for it.

Gerschgorin Circles Applied to Linear Systems

The main defect of the theorems of the last section is their universality, i. e., they prescribe bounds for the entire set of eigenvalues. The theorems of Levy, Hadamard and Gerschgorin provide a very simple geometrical method for determining the bounds on individual eigenvalues. A further advantage of this geometrical

construction is that, with experience, it is possible to tell if a system is stable even though the exact values of the eigenvalues are not known. The theorems mentioned may be combined into one general statement: The eigenvalues of the matrix U lie inside the closed domain G consisting of all circles K_i (i=1, . . . , 2N) with centers U_{ii} and radii r_i where

$$r_i = \sum_{\substack{j=1 \\ \neq i}}^{2N} U_{ij} \tag{2.162}$$

When m circles intersect forming a connected domain H_m precisely m eigenvalues lie in H_m.

Consider the following trivial example of the use of this theorem

$$U_1 = \begin{bmatrix} -2 & -2 & 0 \\ 4\frac{1}{4} & -2 & 0 \\ \frac{1}{2} & 4 & 4 \end{bmatrix} \tag{2.163}$$

It is easy to check that the eigenvalues of U_1 are $\lambda = 4,\ 2+\sqrt{-1}$ and $2-\sqrt{-1}$. Direct application of the theorem leads to a connected domain H_3 composed of 3 circles.

(i) centre at $U_{11} = -2$, radius $|2| = 2$

(ii) centre at $U_{22} = -2$, radius $|4\frac{1}{4}| = 4\frac{1}{4}$ — H_2

(iii) centre at $U_{33} = +4$, radius $|4 + \frac{1}{2}| = 4\frac{1}{2}$ — H_3

Now noting that $S^{-1}US$ has the same eigenvalues as U where S is any non singular matrix it is possible to separate the circles by letting

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \qquad S_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a} \end{bmatrix}$$

Transformations of type S_1 are very useful in applying the theorem to practical examples as they do not change the centres of the circles but may reduce the radius. Applying the transformation S to U it is easy to see that

$$S_1^{-1} U_1 S = \begin{bmatrix} -2 & 2 & 0 \\ 4\frac{1}{4} & -2 & 0 \\ \frac{1}{2a} & \frac{4}{a} & 4 \end{bmatrix}$$

Hence circles (i) and (ii) are unaffected by S_1 but circle (iii) now has radius $9/2a$.

Therefore, by letting $a=9/2$ the radius of circle (iii) is reduced to 1. (In actual fact in this case the radius could be reduced to zero.) Now the system of circles consists of H_2 and the circle of unit radius with centre at 4.

$$U_2 = S_1^{-1} U_1 S_1 = \begin{bmatrix} -2 & 2 & 0 \\ 4\frac{1}{4} & -2 & 0 \\ \frac{1}{9} & \frac{8}{9} & 4 \end{bmatrix}$$

It is possible to reduce the radius of circles (i) and (ii) by letting

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U_3 = S_2^{-1} U_2 S_2 = \begin{bmatrix} -2 & \frac{2}{a} & 0 \\ 4\frac{1}{4}a & -2 & 0 \\ \frac{1}{9} & \frac{8}{9} & 4 \end{bmatrix}$$

Hence by letting $a = 2\sqrt{2/17}$ the circles (i) and (ii) with centre at -2 will have the same radius, $\sqrt{17/2}$. As U_3 may be partitioned into a 2 x 2 symmetric matrix, it is possible to obtain closer bounds on the eigenvalues. However, the main virtue of the use of the Gerschgorin circles is that by a process of successive simplification it is possible to determine quickly, close bounds for any particular root. From the point of view of stability it is easy to see that the system will be stable if it is possible to force all the circles to be in the left half plane. Another technique that is useful in reducing the radius of the Gerschgorin circles is based on the fact that U and U^T have the same set of eigenvalues. Brauer[28] reduced the extent of the domain in the complex plane in which the eigenvalues lie by the introduction of Cassini ovals. However, from the geometrical point of view the construction of circles is simpler. Ostrowski[29] showed that the eigenvalues lie in circles of radii $r_i^{\alpha}\ c_i^{1-\alpha}$

$$0 < \alpha < 1 \quad \text{where} \quad r_i = \sum_{\substack{j=1 \\ \neq i}}^{2N} \left|U_{ij}\right| ; \quad c_i = \sum_{\substack{j=1 \\ \neq i}}^{2N} \left|U_{ji}\right| \qquad (2.164)$$

$$i=1, 2, \ldots, 2N.$$

Generalized Rayleigh Principle for Linear Damped Systems

The main advantage of the Rayleigh method of determining the eigenvalues of a symmetric matrix is that it is possible to set out an iterative scheme of operations which develops a sequence of numbers which converge to some eigenvalue of the matrix. Ostroski[30] has

done extensive work on developing the Rayleigh Quotient Method for non-symmetric matrices. Given the matrix U and the following iterative sequence

$$\xi_\nu = \left[U - \lambda_{\nu-1} I\right]^{-1} \xi_{\nu-1} \qquad \nu = 1, 2, \ldots$$

$$\xi_0 \text{ given}$$

$$\lambda_\nu = \frac{\xi_\nu^* A \xi_\nu}{\xi_\nu^* \xi_\nu}$$

* denote complex transpose

(complex in general) converges to an eigenvalue of the matrix U whose elementary divisors are linear, provided the modulus of the difference between λ_0 and the actual eigenvalue, σ, is less than a certain quantity δ^0 and the norm of the difference vector between the initial vector ξ_0 and the ordinary eigenvector η, associated with σ is less than δ^0 times the norm of η. He also derived results using accelerating convergence techniques for the case of eigenvalues with non linear elementary divisors.

It is interesting to note that by applying Ostrowski's work to the linear damped system with

$$\left[U\right] = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix}$$

the following iterative procedure may be developed.

$$\xi^1_{k+1} = -\left[Q\right]_k \xi^1_k + \frac{1}{(\lambda_k)}\left[Q\right]_k \left[M\right]^{-1}\left[K\right]\xi^2_k$$

$$\xi^2_{k+1} = -\frac{1}{\lambda_{k-1}}\left[Q\right]_k \xi^1_k - \frac{1}{\lambda_k}\left[Q\right]_k \left[M^{-1}C + \lambda_k I\right] \xi^2_k$$

where $\{\xi_k\} = \begin{Bmatrix} \xi^1_k \\ \xi^2_k \end{Bmatrix}$ and ξ^1_o ξ^2_o are given

$$\left[Q\right]_k = \left[\lambda_k I + M^{-1}C + \frac{1}{\lambda_k} M^{-1}K\right]^{-1}$$

$$\lambda_k = \frac{-\xi^{1T}_k M^{-1} C \xi^1_k - \xi^{1T}_k M^{-1} K \xi^2_k + \xi^{2T}_k \xi^1_k}{\xi^{1T}_k \xi^1_k + \xi^{2T}_k \xi^2_k} \qquad (2.165)$$

$$k=0, 1, \ldots$$

Sequence (2.165) may be used to converge on an eigenvalue provided the initial vectors ξ^1_o, ξ^2_o (complex in general) are sufficiently close to an eigenvector of U, associated with a linear elementary divisor. Furthermore, it may be shown that the following equation in a

$$a^2 \phi^* M \phi + a \phi^* C \phi + \phi^* K \phi = 0 \qquad (2.166)$$

has a stationary point when a is an eigenvalue and ϕ is the associated eigenvector, with a linear elementary divisor, of the system specified by M, C and K. When $\xi^1_k \simeq \lambda_k \xi^2_k$ (i.e., ξ^1_k is close to being an eigenvector of the M, C, and K system) it is easy to see that the solution to Eq. (2.166) is approximated by Eq. (2.165). This means that in the case of M, C and K being symmetric matrices it is possible to develop iterative series using only real vectors, for the real and imaginary parts of the eigenvectors and to determine the eigenvalues from Eq. (2.165). The details of these iterative series will not be developed as interest in them is largely academic and they are rather

cumbersome to use for practical calculation.

In conclusion, it must be noted that there are very few really effective methods for obtaining bounds for the eigenvalues of a general matrix. In actual fact, direct application of Rayleigh's method as given by Eqs. (2.148) and (2.154) generally gives closer bounds for passive systems than the other methods discussed above. This is reasonable, in that the matrix U has a rather particular form and bounds developed for general matrices cannot be expected to give very close bounds for matrices of such form. However, although the bounds may be wide, they do give an order of magnitude on the damping and the frequency spectrum of the system.

CHAPTER 3

CONTINUOUS SYSTEMS

Introduction

In this chapter a review of well known results in the analysis of conservative continuous systems is given. Examples of systems which are exactly solvable are presented. The analogy between the analysis of the discrete and continuous system is developed and the idea of generalized viscous damping is introduced. As in the discrete case, continuous systems are divided into two mutually exclusive classes, namely, classical and non-classical systems. Necessary and sufficient conditions for a continuous system to be classical are derived. Finally, it is shown that if the generalized viscous damping term can be expanded into an appropriate infinite series, analogous to the Caughey series in the discrete case, the system is classical.

Theory

The lumped parameter model of the vibration of linear systems generally depends on an idealization of the distribution of the physical characteristics of the system. (e. g., in the mass, spring and dashpot system, the spring has no mass.) In dealing with the vibration of what is normally assumed to be a continuous system (e. g., beam vibrations) it would appear that large scale lumping of the parameters of such a system gives a less accurate description of the motion than the usual partial differential formulation. However, it may be argued that at the microscopic level the difference equation approach (small scale

lumping of parameters) is more meaningful than the usual partial differential equation description. It is not possible to decide such issues here as in the last analysis it is a matter of experimental observation as to which formulation gives the best approximation to the physical reality.

There are two general approaches to the formulation of the equations of motion of continuous systems.[40, 41, 42] The first is to isolate an element of the body, apply Newton's laws to the element and take the limit of the resulting equations as the element shrinks to zero. The other approach is to use the Lagrangian formulation for continuous systems. This latter formulation, which is analogous to the discrete Lagrange's Equations, is obtained by applying Hamilton's variational principle to the system to give the following equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} + \sum_{k=1}^{3} \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta_\ell}{\partial x_k}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \eta_\ell} = F_\ell \qquad \ell = 1, 2, 3 \tag{3.1}$$

where $\mathcal{L}\left(\frac{\partial \eta_i}{\partial x_j}, \dot{\eta}_\ell, x_k, t\right)_{i, j, \ell, k = 1, 2, 3}$ is called the Lagrangian density of the system and has the dimensions of energy/volume.

$$L = \iiint \mathcal{L}\, dx_1\, dx_2\, dx_3$$

is the usual Lagrangian of the system (i.e., L = T-V where T is the total kinetic energy and V the total potential (or strain) energy).

x_k, k=1, 2, 3 are the three continuous indices replacing the

elements of vector $\{x\}$ in the discrete formulation. $\eta_j(x_1, x_2, x_3)$ $j=1, 2, 3$ are the generalized coordinates of the motion along the three axes x_1, x_2 and x_3, respectively. $\dot{\eta}_\ell = \partial\eta_\ell/\partial t$. $F_\ell(\partial\eta_r/\partial x_i, \dot{\eta}_\ell, x_k, t)$ $r, \ell, i, k=1, 2, 3$ is the generalized forcing function associated with η_ℓ, $\ell=1, 2, 3$.

In the theory of elasticity there is a general method,[43] called the Energy Method which is used for the development of the equations of deformation. This method is in fact closely related to the formulation specified by Eq. (3.1), in that the strain energy V, the kinetic energy T and the generalized force F_ℓ are calculated and Hamilton's Principle applied.

As in the discrete case, continuous systems are said to be linear if their partial differential equations of motion are linear. If the system is linear, $\mathcal{L}$ is restricted to be of the form

$$\mathcal{L} = \sum_{n=0}^{2}\sum_{\ell=1}^{3} a_{n\ell}(x, t)\,\dot{\eta}_\ell^{\,n} + \sum_{\substack{m=1\\ m+n\le 2}}^{2}\sum_{n=1}^{2}\sum_{i,j=1}^{3}\sum_{r,s=1}^{3} b_{mn}^{rsij}(x, t)\left(\frac{\partial\eta_i}{\partial x_j}\right)^n\left(\frac{\partial\eta_r}{\partial x_s}\right)^m + \sum_{n=1}^{2}\sum_{\ell=1}^{3} C_{n\ell}(x,t)\,\eta_\ell^{\,n} \qquad (3.2)$$

where $a_{n\ell}(x, t)$, $b_{mn}^{rsij}(x, t)$ and $C_{n\ell}(x, t)$ are continuous functions in x_1, x_2, x_3 and t; the superscripts n and m are power indices and each F_j $j=1, 2, 3$ must be linear in η_j, $\dot{\eta}_j$ and $\ddot{\eta}_j$. There are many systems whose Lagrangian density is not type (3.2). However, for small displacements from the equilibrium position it is possible to approximate

the Lagrangian density by a Taylor series expansion to a form of type (3. 2). This procedure leads to the approximate theories of linear elasticity, linear acoustic theory, etc.

Substituting Eq. (3. 2) for $\mathcal{L}$ in Eq. (3. 1) leads to the equations of motion of general linear continuous systems

$$\frac{d}{dt}\sum_{n=1}^{2} n\, a_{n\ell}(x,t)\,\eta_{\ell_t}^{n-1} + \sum_{k=1}^{3}\frac{d}{dx_k}\left[\frac{\partial}{\partial\left(\frac{\partial\eta_\ell}{\partial x_k}\right)}\left[\sum_{\substack{n=1\\ n+m\le 2}}^{2}\sum_{m=1}^{2}\sum_{i,j=1}^{3}\sum_{r,s=1}^{3} b_{mn}^{rsij}(x,t)\left(\frac{\partial\eta_i}{\partial x_j}\right)^n\left(\frac{\partial\eta_r}{\partial x_s}\right)^m\right]\right] - \sum_{n=1}^{2} n\, C_{n\ell}(x,t)\,\eta_\ell^{n-1} = F_\ell \qquad \ell=1,2,3 \qquad (3.3)$$

It may be seen that it is possible to rewrite Eq. (3. 3) in a more compact form as

$$m_\ell(x,t)\,\eta_{\ell_{tt}}(x,t) + C_\ell(x,t)\,\eta_{\ell_t}(x,t) + \sum_{j=1}^{3} k_{\ell j}(x,t)\,L_{1x}^{\ell j}\,\eta_j(x,t) = F_\ell(x,t) \qquad (3.4)$$

$\ell=1,2,3$

where $L_{1x}^{\ell j}$ is a linear differential operator with respect to the spatial coordinates x_1, x_2 and x_3. $m(x,t)$, $c(x,t)$, $k(x,t)$ and $F(x,t)$ are functions of x_1, x_2, x_3 and t. Equation (3. 4) is the equation of motion of a general continuous linear time varying system. A time invariant system has the following property

$a_{n\ell}(x,t)$, $b_{mn}^{rs\ell k}(x,t)$ and $C_{n\ell}(x,t)$ are functions of $x=(x_1,x_2,x_3)$ alone. (3. 5)

Hence the equations of motion of a general continuous linear time invariant system are

$$m_\ell(x)\ \eta_{\ell_{tt}}(x,t) + \sum_{j=1}^{3} k_{\ell j}(x)\ L_{1x}^{\ell_j}\ \eta_j(x,t) = F_\ell(x,t) \qquad \ell = 1, 2, 3 \qquad (3.6)$$

where $L_{1x}^{\ell_j}$ is a linear differential operator.

Some Common Examples of Linear Time Invariant Continuous Systems

A few common examples[23, 40] of well known linear time invariant continuous systems are given below:

(i) The equation of motion of a vibrating beam may be written as

$$\rho(x)\ u_{tt}(x,t) + \frac{\partial^2}{\partial x^2}\left(E(x)\ I(x)\ \frac{\partial^2 u}{\partial x^2}\right) = f(x,t)$$

where $u(x,t)$ is the transverse motion of a beam with parameters $E(x)$ and $I(x)$. $f(x,t)$ is the externally applied force, e.g., $f(x,t) = -P\frac{\partial^2 u}{\partial x^2}$ if the beam is subjected to an end load P.

(ii) The equation of motion of a plate of constant thickness may be written as

$$c^2 u_{tt}(x,t) + \nabla^4 u = f(x,t)$$

where $u(x,t)$ is transverse motion of the plate, c is a constant and $f(x,t)$ the external force on the plate. If, for example, a rectangular plate is subjected to edge thrusts N_x, N_y and edge shears N_{xy} per unit length

$$f(x,t) = -\frac{N_x}{D}\frac{\partial^2 u}{\partial x^2} - \frac{N_y}{D}\frac{\partial^2 u}{\partial y^2} + \frac{N_{xy}}{D}\frac{\partial^2 u}{\partial x \partial y},$$

where D is a constant.

Damping in Continuous Systems

From Eq. (3. 5) it is seen that a term involving the velocity, $\eta_{\ell t}(x, t)$, can only arise in the equations of motion of a time invariant system through the generalized force term $F_\ell(x, t)$. This is in agreement with the discrete case in which the generalized viscous damping term is introduced into the equations by the generalized force terms. In theoretical physics it is common to introduce the artifice of a mirror-image system[41] so that the formalism of the Lagrangian density and Lagrange's equations may be used with dissipative systems. The idea of the mirror-image system is to consider simultaneously with the system having positive damping a system with negative damping so that the systems taken jointly may be considered as conservative. For instance, for the diffusion equation the Lagrangian density is

$$\mathcal{L} = -(\nabla\psi) \cdot (\nabla\psi^*) - \frac{1}{2} a^2 \left(\psi^* \frac{\partial\psi}{\partial t} - \psi \frac{\partial\psi^*}{\partial t} \right) \tag{3. 7}$$

where ψ is the density of the diffusing fluid, a^2 is the diffusion constant and ψ^* refers to the mirror-image system. By substituting Eq. (3. 7) into Lagrange's Equation (Eq. 3. 1) the following two equations are obtained (assuming ψ and ψ^* are generalized coordinates).

$$\nabla^2 \psi = a^2(\partial\psi/\partial t) \tag{3. 8}$$

$$\nabla^2 \psi^* = -a^2(\partial\psi^*/\partial t) \tag{3. 9}$$

Equation (3. 8) is the well known diffusion equation and Eq. (3. 9) is its mirror image.

In line with the work on discrete systems presented in Chapters I and II, the damping in the continuous systems discussed here will be generalized viscous, i. e., the damping term in the differential equation will be of the type

$$\sum_{j=1}^{3} C_{\ell j}(x)\, L_{2x}^{\ell_j}\, \dot{\eta}_j(x,t) \qquad \ell = 1, 2, 3 \tag{3.10}$$

where $L_{2x}^{\ell_j}$ is a linear differential operator in the spatial coordinates and $C_{\ell j}(x)$ is a function of x alone. The general differential formulation of the linear time invariant continuous system is then

$$m_\ell(x)\, \eta_{\ell_{tt}}(x,t) + \sum_{j=1}^{3} C_{\ell j}(x)\, L_{2x}^{\ell_j}\, \eta_{\ell_t}(x,t) + \sum_{j=1}^{3} k_{\ell j}(x)\, L_{1x}^{\ell_j}\, \eta_\ell(x,t) = F_\ell(x,t) \tag{3.11}$$

Tests on structures and materials[23, 40] lead one to speculate that the damping mechanism in continuous systems is extremely complicated. These tests give very strong indication of non linear and hysteresis damping. However, generalized viscous damping of form (3.10) is directly analogous to viscous damping in discrete problems and it is interesting to determine what effect such damping has on the response of linear continuous systems. Moreover, for small displacements the response of a system with only slightly non linear or hysteresis type damping may be approximated by an equivalent linear or bilinear system. Furthermore, it should be remarked[41] that very little work has been done by mathematicians on non linear partial differential equations. This is so because of the inherent difficulty in the analysis

of such equations and their comparative rarity in classical physics.

Solution of the Undamped Continuous System Using the Differential Formulation

The equations of motion of the undamped linear continuous system may be written as

$$m(x)\, u_{tt}(x, t) + k(x) L_{1x}\, u(x, t) = F(x, t) \tag{3.12}$$

where $x = (x_1, x_2, x_3)$ represents the spatial coordinates

$$u(x, t) = \begin{Bmatrix} u_1(x, t) \\ u_2(x, t) \\ u_3(x, t) \end{Bmatrix} \quad ; \qquad F(x, t) = \begin{Bmatrix} F_1(x, t) \\ F_2(x, t) \\ F_3(x, t) \end{Bmatrix}$$

$$m(x) = \begin{bmatrix} m_1(x) & 0 & 0 \\ 0 & m_2(x) & 0 \\ 0 & 0 & m_3(x) \end{bmatrix} \quad ; \quad k(x) L_{1x} = \begin{bmatrix} k_{11}(x) L_{1x}^{11} & k_{12}(x) L_{1x}^{12} & k_{13}(x) L_{1x}^{13} \\ . & k_{22}(x) L_{1x}^{22} & . \\ . & k_{32}(x) L_{1x}^{32} & k_{33}(x) L_{1x}^{33} \end{bmatrix}$$

$u_i(x, t)$, $i=1, 2, 3$, are the generalized coordinates specifying the displacement of the medium.

The two general methods[41, 42, 45, 46, 47] of solution of partial differential equations of type (3.12) are the integral solution and the separated solution. The integral solution has the advantage of generality for usually the integral is invariant under coordinate transformation and once the kernel of the integral (or equivalently the Green's Function) is known, the solution to the homogeneous or inhomogeneous problem may be determined -- at least in principle. However, the integral solution is not always the most satisfactory

solution as the integral may not be readily integratable in closed form and recourse must be had to numerical integration. The second method of solving linear partial differential equations is the method of separation of the original partial differential equation into a set of ordinary differential equations each involving only one variable. This technique does not have the universality of the integral solution approach but if it can be used it generally leads to a simpler analysis, in that the solution of ordinary differential equations is well understood. However, it must be borne in mind that not all coordinate systems will allow separation of variables and that there exists some linear partial differential equations which simply cannot be separated in any coordinate system. For a partial differential equation to be separable in a particular coordinate system, the nodal surfaces ($u(x,t) \equiv 0$ for a given boundary surface and variable relationships between the generalized coordinates $u(x,t)$ on the boundary, e.g., X is the boundary surface $u(X) = a$; $\partial u/\partial \eta\big|_X = b$ where a and b are constants and η is the normal to X, each a and b generate one nodal surface $u(x,t)=0$) must coincide with the coordinate surfaces.

Provided the partial differential equation is separable in a given coordinate system (with respect to some boundary surface) the resulting ordinary linear differential equations may be solved by any of the standard methods. It is important to note that if the equation is separable then linear combinations of the products of the solutions of the resulting ordinary differential equations give all possible solutions to the original problem.

In the case of the vibration of the undamped continuous system it is always possible to separate the time and the spatial parts of the complete solutions.

Let

$$u(x, t) = X(x)\ T(t) \tag{3.13}$$

On substituting Eq. (3.13) into (3.12)

$$m(x)\ X(x)\ \ddot{T}(t) + k(x)L_{1x}X(x)\ T(t) = F(x, t) \tag{3.14}$$

To solve Eq. (3.14), the homogeneous problem ($F(x, t) \equiv 0$) must first be solved. Substituting from Eq. (3.11), it is easy to see that the homogeneous part of Eq. (3.14) may be expanded as follows

$$\begin{bmatrix} m_1(x)\ X_1(x) & 0 & 0 \\ 0 & m_2(x)\ X_2(x) & 0 \\ 0 & 0 & m_3(x)X_3(x) \end{bmatrix} \begin{Bmatrix} \ddot{T}_1(t) \\ \ddot{T}_2(t) \\ \ddot{T}_3(t) \end{Bmatrix}$$

$$+ \begin{bmatrix} k_{11}(x)L_{1x}^{11}X_1(x) & k_{12}(x)L_{1x}^{12}X_2(x) & k_{13}(x)L_{1x}^{13}X_3(x) \\ . & k_{22}(x)L_{1x}^{22}X_2(x) & . \\ . & . & k_{33}(x)L_{1x}^{33}X_3(x) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.15}$$

where

$$X(x)\ T(t) = \begin{Bmatrix} X_1(x)\ T_1(t) \\ X_2(x)\ T_2(t) \\ X_3(x)\ T_3(t) \end{Bmatrix}$$

or to simplify the notation

$$\left[M(x)\right] \left\{\ddot{T}(t)\right\} + \left[K(x)\right] \left\{T(t)\right\} = 0$$

where

$$\left[M(x)\right] = \begin{bmatrix} m_1(x)\,X_1(x) & 0 & 0 \\ 0 & m_2(x)X_2(x) & 0 \\ 0 & 0 & m_3(x)\,X_3(x) \end{bmatrix};$$

$$\left[K(x)\right] = \begin{bmatrix} k_{11}(x)L_{1x}^{11}X_1(x) & \cdot & k_{13}(x)L_{1x}^{13}X_3(x) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & k_{33}(x)L_{1x}^{33}X_3(x) \end{bmatrix}$$

$$\left[M(x)\right]^{-1} = \begin{bmatrix} \left\{m_1(x)X_1(x)\right\}^{-1} & 0 & 0 \\ 0 & \left\{m_2(x)X_2(x)\right\}^{-1} & 0 \\ 0 & 0 & \left\{m_3(x)X_3(x)\right\}^{-1} \end{bmatrix}$$

$$T(t) = \begin{Bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{Bmatrix} \tag{3.16}$$

Premultiplying Eq. (3.16) by $\left[M(x)\right]^{-1}$

$$\left\{\ddot{T}(t)\right\} + \left[M(x)\right]^{-1}\left[K(x)\right]\left\{T(t)\right\} = 0 \tag{3.17}$$

In Eq. (3.17) let

$$\begin{aligned} \left[A(x)\right] &= \left[M(x)\right]^{-1}\left[K(x)\right] \\ \left[B(x)\right]\left[B(x)\right] &= -\left[A(x)\right] \end{aligned} \tag{3.18}$$

$$\therefore \left\{T(t)\right\} = e^{\left[B(x)\right]t}\left\{T(0)\right\} \quad \text{for all } x, t \tag{3.19}$$

where $\left\{T(0)\right\} = \left\{T(t)\right\}$ at $t = 0$

Now T(t) is a vector with elements which are functions of t only.

Letting

$$\left\{T(0)\right\}_{\ell} = \begin{Bmatrix} \delta_{1\ell} \\ \delta_{2\ell} \\ \delta_{3\ell} \end{Bmatrix} \qquad \ell = 1, 2, 3 \qquad \delta_{ij} = 1 \quad i = j \qquad = 0 \quad i \neq j$$

Equation (3.18) may be expanded

$$\left\{T(t)\right\}_{\ell} = \left[I + t\left[B(x)\right] + \frac{t^2}{2}\left[B(x)\right]^2 + \ldots\right] \begin{Bmatrix} \delta_{1\ell} \\ \delta_{2\ell} \\ \delta_{3\ell} \end{Bmatrix} \tag{3.20}$$

$$= \begin{Bmatrix} \delta_{1\ell} \\ \delta_{2\ell} \\ \delta_{3\ell} \end{Bmatrix} + t\begin{Bmatrix} B^1_{11}(x)\,\delta_{1\ell} + B^1_{12}(x)\,\delta_{2\ell} + B^1_{13}(x)\,\delta_{3\ell} \\ B^1_{21}(x)\,\delta_{1\ell} + B^1_{22}(x)\,\delta_{2\ell} + B^1_{23}(x)\,\delta_{3\ell} \\ B^1_{31}(x)\,\delta_{1\ell} + B^1_{32}(x)\,\delta_{2\ell} + B^1_{33}(x)\,\delta_{3\ell} \end{Bmatrix} + \ldots \tag{3.21}$$

where B^n_{ij} is the ij^{th} element of $\left[B(x)\right]^n$.

As 1, t, t^2----t^n---- are a linearly independent set of functions of t, the elements of each of the vectors in expression (3.21) must be constants if $\left\{T(t)\right\}$ is to be a vector whose elements are functions of t alone. By letting $\ell = 1$, 2, and 3, it is easy to see that the matrix $\left[B(x)\right]$ must be a matrix with constant elements. From Eq. (3.18) it may also be seen that

$$\left[A(x)\right] = \left[C\right]$$

where $\left[C\right]$ is a 3 x 3 matrix with constant elements. Therefore, in the undamped case a solution of type given by (3.13) leads to two differential systems of equations

$$\left[A(x)\right] = \left[C\right]$$

$$\{\ddot{T}\} = -[C]\{T\} \tag{3.22}$$

where $[C]$ is a 3 x 3 matrix with constant elements. In (3.22), it should be noted that $[A(x)]$ is a matrix whose elements are functions of $u_i(x)$.

Although it has been shown that the undamped problem is separable into spatial and time differential systems of equations, it cannot be concluded that the spatial system of equations can be further separated into a system of equations, each member of which is an ordinary differential equation in one of the three generalized coordinates.

If the continuous vibrating problem involves only one spatial coordinate (e. g., the Euler beam vibration problem) the system can always be separated into two ordinary differential equations, one in x and the other in t. The separation constants of all linear partial differential equations are determined by the boundary conditions, e. g., periodicity or continuity requirements may restrict the allowable values of the separation constants. Although the final solution is made up of a series of terms, each one of which is a separable solution, it is not itself separable. Separation of solutions in two dimensions is particularly simple for three reasons. In the first case, there is only one separation constant, so that the factored solutions form a one parameter family, which makes the fitting of boundary conditions by the series solution fairly simple. Secondly, in the two dimensional problems the conditions for separation are simple, e. g., the following partial differential equation in u and v

$$\frac{\partial^2 X(u, v)}{\partial u^2} + \frac{\partial^2 X(u, v)}{\partial v^2} + f(u, v)\, X(u, v) = 0 \qquad (3.23)$$

is separable (under suitable boundary conditions) if and only if

$$f(u, v) = a\, g(u) + b\, k(v)$$

where a and b are constants, and g(u) and k(v) are functions of u and v respectively. Finally in the two dimensional case the condition that the nodal surfaces and the coordinate surfaces coincide is both a necessary and sufficient for separation.

Turning to the higher dimensional cases (i. e., three or four dimensions) the situation is considerably more complicated. In the first place there are two or three separation constants. Each of the separated equations may depend on more than one constant, which would make the satisfying of boundary conditions extremely complicated Generally, however, the situation is simpler in that one or two of the separated equations contain only one separation constant.

In some coordinate systems the solution of three or four dimensional problems are not completely separable,[41] i. e., the solution takes the form $R(x_1, x_2, x_3)\, X_1(x_1)\, X_2(x_2)\, X_3(x_3)$ where the function $R(x_1, x_2, x_3)$ is taken outside the usual sum over the allowed values of the separation constants. Moreover these problems are generally more difficult than those problems with complete separation. Hence, the condition of the coinciding of the nodal surfaces and the coordinate surfaces is a necessary though not a sufficient condition for separability of the higher dimensional problems.

The most general analytical tool available to determine separability of partial differential equations in a particular coordinate system under suitable boundary conditions is the so-called Stäckel Determinant. However, except for simple well known equations, e. g., the wave equation, the Stäckel Determinant is rarely used in practical work.

Separability of the Time and Spatial Problems in Damped Systems

The equations of motion of viscously damped continuous systems may be written as

$$m(x)\, u_{tt}(x,t) + c(x)L_2\, u_t(x,t) + k(x)L_{1x}\, u(x,t) = F(x,t) \qquad (3.24)$$

where $x = (x_1, x_2, x_3)$

$$u(x,t) = \begin{Bmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{Bmatrix}; \quad F(x,t) = \begin{Bmatrix} F_1(x,t) \\ F_2(x,t) \\ F_3(x,t) \end{Bmatrix}$$

$$k(x)L_{1x} = \begin{bmatrix} k_{11}(x)L_{1x}^{11} & k_{12}(x)L_{1x}^{12} & k_{13}(x)L_{1x}^{13} \\ . & k_{22}(x)L_{1x}^{22} & . \\ . & . & k_{33}(x)L_{1x}^{22} \end{bmatrix}$$

$$m(x) = \begin{bmatrix} m_1(x) & 0 & 0 \\ 0 & m_2(x) & 0 \\ 0 & 0 & m_3(x) \end{bmatrix}; c(x)L_{2x} = \begin{bmatrix} c_{11}(x)L_{1x}^{11} & c_{12}(x)L_{2x}^{12} & c_{13}(x)L_{3x}^{13} \\ . & c_{22}(x)L_{2x}^{22} & c_{23}(x)L_{2x}^{23} \\ . & . & c_{33}(x)L_{2x}^{33} \end{bmatrix}$$

On adding to Eq. (3.24) the identity

$$m(x)\ \ p(x,t) - m(x)\ u_t(x,t) = 0 \qquad (3.25)$$

where $p(x,t) = u_t(x,t)$

the continuous system may be described by the following system equations

$$\left.\begin{aligned} m(x)\ p_t(x,t) + c(x)\, L_{2x} p(x,t) + k(x) L_{1x}\ u(x,t) &= F(x,t) \\ m(x)\ p(x,t) - m(x)\ p(x,t) &= 0 \end{aligned}\right\}$$

or

$$n(x)\ s_t(x,t) + r(x) L_x\ s(x,t) = F$$

where

$$s(x,t) = \begin{Bmatrix} u_{1t}(x,t) \\ u_{2t}(x,t) \\ u_{3t}(x,t) \\ u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{Bmatrix}; \quad n(x) = \begin{bmatrix} m_1(x) & 0 & 0 & 0 & . & . \\ . & m_2(x) & 0 & . & . & . \\ . & . & m_3(x) & 0 & . & . \\ . & . & . & m_1(x) & 0 & 0 \\ . & . & . & . & m_2(x) & 0 \\ . & . & . & . & . & m_3(x) \end{bmatrix}$$

$$r(x)L_x = \begin{bmatrix} c_{11}(x)L_{2x}^{11} & . & c_{13}(x)L_{2x}^{13} & k_{11}(x)L_{1x}^{11} & . & k_{13}(x)L_{1x}^{13} \\ . & c_{21}(x)L_{2x}^{21} & . & . & k_{22}(x)L_{1x}^{22} & . \\ c_{31}(x)L_{2x}^{31} & . & c_{33}(x)L_{2x}^{33} & k_{31}(x)L_{1x}^{31} & . & k_{33}(x)L_{1x}^{33} \\ -m_1(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & -m_2(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & -m_3(x) & 0 & 0 & 0 \end{bmatrix}$$

(3.26)

$$F = \begin{Bmatrix} F_1(x, t) \\ F_2(x, t) \\ F_3(x, t) \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Now Eq. (3.26) has a form similar to Eq. (3.15) and therefore the homogeneous form of Eq. (3.26) is separable as far as time and spatial coordinates are concerned. Hence the damped continuous system is at least separable into time and spatial problems.

As is usual in linear differential equations the complete solution to the inhomogeneous problem is composed of the sum of the solution to the homogeneous problem with the given boundary conditions and the solution to the inhomogeneous problem with zero boundary conditions.

Solutions of the Separable Equations Derived From the Partial Differential Equations of Motion of Continuous Systems

Provided the system of equations specified by Eq. (3.24) can be separated into four ordinary differential equations (in t, x_1, x_2 and x_3) there are well known techniques[41] available for completing the determination of the solution. Reduction to quadratures by means of an integrating factor is generally used for first order differential equations and may sometimes be used for higher order problems. Another technique is the series solution the basis for which is found in complex variable theory. In many cases an integral representation of the solution is more useful than the set of series solutions (one series

solution about each singular point of the differential equation) in that the form of the series solution changes over the complex plane, depending on the nearest singularity. The integral representation, if it exists, has no problems of convergence as does the series solution, but the actual evaluation of the integral at some point of interest may be far from trivial. The kernel of the integral representation is to some extent arbitrary but the more usual integral representations are nothing more than the well known inverse integral transforms, e. g., Laplace, Euler or Fourier.

In the literature, a great amount of work has been done on the solution of special types of partial differential equations under particular types of boundary conditions. The process of fitting the boundary conditions generally involves the selection of a particular set of separation constants. As far as the solution of the separated equations themselves are concerned, the separation constants are in general quite arbitrary and it is only when particular boundary conditions are involved that the separation constants must take on a particular set of values.

Solution of the Separated Equations in Terms of Eigenfunctions

In the last section, on the solution of the separated equations (which are ordinary differential equations in one variable) the solution by means of a power series about singular points of the differential equation was discussed. Such a series is, in a sense, a special case

of a more general technique known as the determination of the solution as an eigenfunction expansion. Although, in principle it is possible to discuss eigenfunctions in n, $(n > 1)$, dimensions, the practical determination of eigenfunctions has been largely restricted to ordinary differential equations in one variable. Therefore, unless the partial differential equation is completely separable, i. e., into ordinary differential equations, the eigenfunction approach has only theoretical interest. Furthermore, it is well to note that even if the solution of the original partial differential equation has been reduced to a set of eigenfunction-eigenvalue equations, the actual determination of the eigenfunctions is not easy. In the majority of cases which are of interest in the applied fields, the eigenfunctions must be obtained by numerical methods using a digital computer. In the eigenfunction expansion of the solution, it is the boundary conditions which determine the eigenvalues, which now take the place of the separation constants. The determination of a set of eigenfunctions is nothing more than the solution of a two point boundary value problem. If the end points are singular points of the differential equation, the boundary condition may be simply that the solution remain finite there. If the end points are ordinary points, the boundary conditions may be homogeneous, i. e., the ratio between the value and the slope of the function equal a certain constant or that the solution be periodic with a certain period, etc.

One of the main problems with an eigenfunction expansion of the solution is the question of completeness of the set of eigenfunctions.

It is obvious that if a denumerable infinity of eigenfunctions are to represent a function in function space some restrictions are required on the function. Such restrictions in effect separate out a subspace containing all functions which are continuous except at a finite number of discontinuities from the nondenumerably infinite-dimensional space. There is a close relationship between the eigenfunction theory of linear continuous systems and the normal mode theory of linear discrete systems. However, the theory of eigenfunction expansions is not as well developed as the theory of the eigenvector expansion in N dimensional space. Basically the difficulty is that although a spectral theory of general linear operators exists, its application to concrete problems in differential and integral equations is quite difficult. In the discrete case, the ordinary and generalized eigenvectors provided a basis for the space which greatly simplified the analysis. Unfortunately, in the continuous system, although an analogous basis exists its use is greatly restricted by the difficulties of computation. Therefore, at this point, it is proposed to limit the discussion to a subclass of continuous vibrating systems, namely those systems whose operators are self-adjoint. Such systems include as special cases of the well known continuous undamped vibration problems discussed in the literature.

Theory of Linear Self-Adjoint Differential Systems

A complete definition of a linear differential operator L_1 involves the specification of the linear vector space S of functions on which the

operator acts. In the present work only differential equations over a finite interval will be considered. The first requirement for S is that all functions in it are real valued and Lebesque square integrable over the finite interval of interest (here normalized to $0 \rightarrow 1$ for convenience). Next the end conditions must be considered for a complete definition of L_1. The end conditions, of which there are a linearly independent set equal in number to the highest order derivative in the differential operator (n), must be satisfied by all functions in S. Finally, only those functions which are Lebesque square integrable, satisfy the boundary conditions and possess a piecewise continuous n^{th} derivative are in S. Such a set of functions is known as the domain of definition or the manifold of L_1.

The adjoint of a differential operator L_1 is defined by considering

$$\langle v, L_1 u \rangle = \langle w, u \rangle = \int_0^1 w(x)u(x)dx \tag{3.27}$$

and putting $w = L_1^* v$ where $\langle , \rangle$ indicates the scalar product in the space S. To obtain $\langle w, u \rangle$ it is necessary to integrate $\langle v, L_1 u \rangle$ by parts, e. g., suppose $L_1 = d/dx$ and the boundary condition is $u(0) = 4u(1)$

$$\langle v, L_1 u \rangle = \int_0^1 v \frac{du}{dx} dx = vu \Big|_0^1 - \int_0^1 u \frac{dv}{dx} dx$$

$$= u(0)\left[\frac{1}{4} v(1) - v(0)\right] - \int_0^1 u \frac{dv}{dx} dx = \langle w, u \rangle \tag{3.28}$$

$\therefore L_1^*$ consists of 2 parts; a differential operator -d/dx and boundary conditions. The adjoint to d/dx with the boundary conditions u(0) = 4u(1) is the operator -d/dx with the boundary conditions $u(0) = \frac{1}{4} u(1)$. If $L_1^* = L_1$ then the differential operator is formally self adjoint; if, moreover, the boundary conditions for L_1^* and L_1 are identical the operator is said to be Self Adjoint or Hermitian. There is an obvious connection between self adjoint differential operators and Hermitian matrices. The boundary conditions in the problems of interest in this work are homogeneous and linear, i.e., for a second order operator, the most general boundary conditions are

$$\left.\begin{aligned} B_1(u):\ & a_1 u(1) + a_1' \frac{du}{dx}\bigg|_{x=1} + \beta_1 u(0) + \beta_1' \frac{du}{dx}\bigg|_{x=0} = 0 \\ B_2(u):\ & a_2 u(1) + a_2' \frac{du}{dx}\bigg|_{x=1} + \beta_2 u(0) + \beta_2' \frac{du}{dx}\bigg|_{x=0} = 0 \end{aligned}\right\} \qquad (3.29)$$

a_1, a_1', a_2, a_2', β_1, β_1', β_2 and β_2' are constants

Two well known differential operators

$$L_{1x} = \frac{d}{dx} a(x) \frac{d}{dx} \ ; \qquad L_{2x} = \frac{d^2}{dx^2} b(x) \frac{d^2}{dx^2} \qquad (3.30)$$

occur very often in the separated equations of motion of continuous systems. They are both self adjoint operators in a suitable domain of definition, e.g., L_{1x} is self adjoint provided

$$\langle v, L_{1x} u \rangle = \langle u, L_{1x} v \rangle \qquad (3.31)$$

where u(x), v(x) are in the same manifold (i.e., are functions which satisfy boundary conditions of type (3.29)). But

$$\langle v, L_{1x} u \rangle = \int_0^1 v(x) \frac{d}{dx} \left(a(x) \frac{du}{dx} \right) dx = v(x)\, a(x) \frac{du}{dx} \Big|_0^1 - \int a(x) \frac{du}{dx} \frac{dv}{dx}\, dx$$

$$= \left[v(x)\, a(x) \frac{du}{dx} - u(x)\, a(x) \frac{dv}{dx} \right] \Big|_0^1 + \int_0^1 u(x) \frac{d}{dx} \left(a(x) \frac{dv}{dx} \right) dx \qquad (3.32)$$

$\therefore L_{1x}$ is self adjoint if

$$\left[v(x)\, a(x) \frac{du}{dx} - u(x)\, a(x) \frac{dv}{dx} \right] \Big|_0^1 = 0 \qquad (3.33)$$

where v(x) and u(x) are in the same manifold. Equation (3.33) can be satisfied under fairly general conditions, but it is always satisfied if the boundary conditions of Eq. (3.29) are unmixed, i.e., $B_1(u)$ and $B_2(u)$ only contain conditions at either x = 1 or x = 0 but not at both. Likewise if the boundary conditions are periodic in the form

$$u(0) = u(1); \qquad u'(0) = u'(1) \qquad (3.34)$$

Eq. (3.33) is satisfied.

The eigenfunctions (u_i) and the eigenvalues (λ_i) of a differential operator L_{1x} are defined as follows

$$L_{1x}\, u_i(x) = \lambda_i u_i(x) \qquad i=1, 2, \ldots \qquad (3.35)$$

where u_i satisfy the boundary conditions of the domain of influence of L_{1x}. The completeness of the eigenfunction expansions to be used in the rest of this chapter depends on the following statement.[45, 47, 48]

A boundary value problem of the following type

$$L_{1x}\, u(x) + \lambda\, s(x)\, u(x) = 0 \qquad (3.36)$$

where L_{1x} is a regular self-adjoint operator defined over a finite

domain in which s(x) is positive, or zero at most a finite number of points, possesses a complete set of orthonormal eigenfunctions. This means that every function $\omega(x)$ which satisfies the boundary conditions of the domain of definition of L_{1x} and has continuous first, second, ... $(n-1)^{th}$ derivatives and piecewise continuous n^{th} derivative where n is the highest derivative in the expanded operator L_{1x}, can be represented in an absolutely and uniformly convergent (in the mean sense) series

$$\omega(x) = \sum_{n=1}^{\infty} c_n u_n(x); \qquad ; \qquad c_n \text{ are constants}$$

There is a class of problems, known in the literature as Sturm-Liouville problems,[45, 46, 47, 48] which arise very frequently in the solution of vibration problems by the separation of variables. The general Sturm-Liouville problem may be written as

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] - q(x)u + \lambda s(x)u = 0 \qquad (3.37)$$

under boundary conditions which makes the operator $\left[\frac{d}{dx}(p(x)\frac{d}{dx}) - q(x)\right]$ self-adjoint. If the operator acts on functions defined on a closed finite interval and p(x) and s(x) are different from zero in this interval, the system described by Eq. (3.37) is called a regular Sturm-Liouville system. If the interval is infinite or p(x) or s(x) are zero in a finite interval, such systems are called singular Sturm-Liouville systems. In Sturm-Liouville problems, the inner product is defined using the weight function s(x). Sturm-Liouville systems, as a subclass of self adjoint systems, are particularly easy to work with in that there are many computational aids available for use with such systems. As

an example, Sturm's Comparison theorem depicts the position of the nodes of the eigenfunctions and shows that the n^{th} eigenfunction of a regular Sturm-Liouville problem has exactly n nodes. There are various asymptotic approximations for both the eigenvalues and eigenfunctions of regular Sturm-Liouville systems. The eigenfunctions of singular Sturm-Liouville can be complete (e. g., Bessel functions) and are orthogonal provided they are square-integrable with a suitable weighting function over the interval of definition.

Self Adjoint Undamped Systems

The equations of motion of a self adjoint undamped vibrating system may be written as

$$p(x)\, u_{tt}(x, t) + \left[L_{1x}\right] u(x, t) = F(x, t) \tag{3. 38}$$

where in the most general case $p(x)$, $u(x, t)$ and $F(x, t)$ are vector quantities with functional elements, x the spatial coordinate represents x_1, x_2, x_3 and $\left[L_{1x}\right]$ is a matrix linear differential self adjoint operator, i. e.,

$$\langle v, \left[L_{1x}\right] u\rangle = \langle u, \left[L_{1x}\right] v\rangle \tag{3. 39}$$

the vectors u and v satisfying prescribed boundary conditions where the inner product sign $\langle , \rangle$ is interpreted as the sum of the inner products (with a suitable weighting function) of the individual vector components. Since the general formulation is no more difficult, theoretically, than the simpler two dimensional partial differential equation formulation, the latter will be used to simplify the notation.

The equations of motion of a one dimensional self adjoint continuous vibrating system may be written as

$$p(x)\, u_{tt}(x, t) + L'_{1x}\, u(x, t) = f_1(x, t) \tag{3.40}$$

where x is the one dimensional spatial coordinate. L'_{1x} is a self adjoint linear differential operator which has its own manifold, the elements of which satisfy the boundary and interval conditions of Eq. (3.40).

In the well known vibration problems, e.g., string vibrations, p(x) is the distributed mass, f(x, t) is the time and space varying exciting force and L_{1x} is the stiffness operator. Equation (3.40) may be written in canonical form by use of the following transformation

$$z(x, t) = \frac{1}{\sqrt{p(x)}}\, u(x, t); \qquad p(x) > 0 \tag{3.41}$$

$$0 \leq x \leq 1$$

On substituting Eq. (3.41) into Eq. (3.40)

$$\sqrt{p(x)}\, z_{tt}(x, t) + L'_{1x}\, \frac{1}{\sqrt{p(x)}}\, z(x, t) = f_1(x, t) \tag{3.42}$$

Multiplying both sides of Eq. (3.42) by $1/\sqrt{p(x)}$

$$z_{tt}(x, t) + \frac{1}{\sqrt{p(x)}}\, L'_{1x}\, \frac{1}{\sqrt{p(x)}}\, z(x, t) = \frac{1}{\sqrt{p(x)}}\, f_1(x, t) \tag{3.43}$$

Now as L'_{1x} is self adjoint in the domain of Eq. (3.40)

$$\langle u,\ L'_{1x}\, v \rangle = \langle v,\ L'_{1x}\, u \rangle$$

where u, v satisfy the boundary and interval conditions of Eq. (3.40).

$$\therefore \left\langle \sqrt{p(x)}\, u, \frac{1}{\sqrt{p(x)}} L'_{1x} \frac{1}{\sqrt{p(x)}} \sqrt{p(x)}\, v \right\rangle = \left\langle \sqrt{p(x)}\, v, \frac{1}{\sqrt{p(x)}} L'_{1x} \frac{1}{\sqrt{p(x)}} \sqrt{p(x)}\, u \right\rangle \tag{3.44}$$

Hence $1/\sqrt{p(x)}\; L'_{1x}\; 1/\sqrt{p(x)}$ is self adjoint in the domain of the transformation (3.41) where $u(x, t)$ is a function in the domain of definition of L'_{1x}. Therefore the canonical form of Eq. (3.40) as given by Eq. (3.43) possesses a self adjoint operator. Letting

$$\frac{1}{\sqrt{p(x)}} L'_{1x} \frac{1}{\sqrt{p(x)}} = L_{1x} \tag{3.45}$$

and

$$\frac{1}{\sqrt{p(x)}} f_1(x, t) = f(x, t)$$

the canonical form of Eq. (3.40) may be written as

$$z_{tt}(x, t) + L_{1x}\, z(x, t) = f(x, t) \tag{3.46}$$

where L_{1x} is a self adjoint operator.

It is well to note that all second order systems may be transformed to a formally self adjoint system. Consider

$$y_{tt} + L_x y = y_{tt} + a(x) \frac{d^2 y}{dx^2} + b(x) \frac{dy}{dx} + c(x)\, y = f(x, t) \tag{3.47}$$

$$a(x) > 0 \qquad 0 \leq x \leq 1$$

where

$$L_x = \left\{ a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x) \right\}$$

under the following boundary conditions

$$B_1(y):\ \alpha_1\, y(0) + \alpha_1' \left.\frac{dy}{dx}\right|_{x=0} + \beta_1\, y(1) + \beta_1' \left.\frac{dy}{dx}\right|_{x=1} = 0$$

$$B_2(y):\ \alpha_2\, y(0) + \alpha_2' \left.\frac{dy}{dx}\right|_{x=0} + \beta_2\, y(1) + \beta_2' \left.\frac{dy}{dx}\right|_{x=1} = 0$$

The formal adjoint of L_x may be easily seen to be

$$L_x^* = \left\{\frac{d^2}{dx^2}\, a(x) - \frac{d}{dx}\, b(x) + c(x)\right\}$$

$$= \left\{ a(x)\frac{d^2}{dx^2} + (2a'(x)-b(x))\frac{d}{dx} + (c(x)+a''(x)-b'(x)) \right\}$$

where ' indicates differentiation with respect to x. Now if both sides of Eq. (3.47) are multiplied by g(x) where

$$g(x)\left(\frac{b(x)-a'(x)}{a(x)}\right) = g'(x)$$

or

$$g(x) = \exp \int_0^x \frac{b(x)-a'(x)}{a(x)}\, dx \tag{3.49}$$

the original non formally self adjoint system is now reduced to the formally self adjoint system

$$g(x)\, y_{tt}(x,t) + L_{1x} y(x,t) = g(x)\, f(x,t)$$

where

$$L_{1x} = \left\{\frac{d}{dx}\, g(x)\, a(x)\, \frac{d}{dx} + g(x)\, c(x)\right\} \tag{3.50}$$

is a formally self adjoint operator. The question of reducing a non self adjoint second order differential equation to a self adjoint system depends solely on the boundary conditions. For equations of higher order there is no general method of reducing them to a formally self

adjoint system.

Returning to the solution of Eq. (3.46), first solve the homogeneous problem

$$z_{tt}(x,t) + L_{1x}\, z(x,t) = 0 \tag{3.51}$$

As L_{1x} is a self adjoint operator there exists an infinite sequence of orthogonal eigenfunctions $\phi_i(x)$ with associated real eigenvalues λ_i. As the ϕ_i, i=1, 2, ... are complete, z(x, t) may be expanded as follows. Let

$$z(x,t) = \sum_i a_i(t)\, \phi_i(x) \tag{3.52}$$

On substituting Eq. (3.52) into (3.51)

$$\sum_i \ddot{a}_i(t)\ \phi_i(x) + \sum_i a_i(t)\, L_1\, \phi_i(x) = 0 \tag{3.53}$$

But $L_1 \phi_i(x) = \lambda_i\, \phi_i(x)$.

$$\therefore \qquad \sum_i \ddot{a}_i(t)\, \phi_i(x) + \sum_i a_i(t)\, \lambda_i\, \phi_i(x) = 0 \tag{3.54}$$

On multiplying both sides of Eq. (3.54) by $\phi_j(x)$ and integrating each term over x between 0 and 1,

$$\sum_i \int_0^1 \ddot{a}_i(t)\, \phi_i(x)\, \phi_j(x)\, dx + \sum_i \int_0^1 a_i(t)\, \lambda_i\, \phi_i(x)\, \phi_j(x)\, dx = 0 \tag{3.55}$$

But due to the orthonormalization of the eigenfunctions, Eq. (3.55) may be reduced to

$$\ddot{a}_j(t) + \lambda_j\, a_j(t) = 0 \qquad j=1, 2, \ldots \tag{3.56}$$

$$\therefore \qquad a_j(t) = A_j \cos(\sqrt{\lambda_j}\, t) + B_j \sin(\sqrt{\lambda_j}\, t) \qquad j=1, 2, \ldots \qquad (3.57)$$

Hence the solution to the homogeneous problem, Eq. (3.51), is

$$z(x,t) = \sum_i (A_i \cos(\sqrt{\lambda_i}\, t)\phi_i(x) + B_i \sin(\sqrt{\lambda_i}\, t)\phi_i(x)) \qquad (3.58)$$

A_i and B_i must be determined from initial conditions

$$\begin{aligned} z(x,0) &= z_o(x) \\ z_t(x,0) &= \dot{z}_o(x) \end{aligned} \qquad (3.59)$$

From Eq. (3.58)

$$\begin{aligned} z_o(x) &= \sum_i A_i \phi_i(x) \\ \dot{z}_{ot}(x) &= \sum_i B_i \sqrt{\lambda_i}\, \phi_i(x) \end{aligned} \qquad (3.60)$$

$\therefore$ using the orthonormalization conditions of the eigenfunctions it may be seen that

$$\begin{aligned} A_i &= \int_0^1 z_o(x)\, \phi_i(x)\, dx \\ B_i &= \frac{1}{\sqrt{\lambda_i}} \int_0^1 \dot{z}_o(x)\, \phi_i(x)\, dx \qquad \lambda_i \neq 0 \end{aligned} \qquad (3.61)$$

$i=1, 2, \ldots$

The complete solution to the self adjoint system of Eq. (3.46) involves the sum of the solution to the homogeneous problem (here given by Eqs. (3.58) and (3.61)) and a particular solution with zero initial conditions to the inhomogeneous problem. The particular

solution may be easily derived as follows. Let

$$z(x,t) = \sum_i b_i(t)\, \phi_i(x) \tag{3.62}$$

Substituting Eq. (3.62) into Eq. (3.47), multiplying both sides of the resulting equation by $\phi_j(x)$ and using the orthonormalization property of the eigenfunctions,

$$b_j(t) + \lambda_j\, b_j(t) = \int_0^1 f(x,t)\, \phi_j(x)\, dx = g_j(t)$$

$$\therefore\ b_j(t) = \int_0^t g_j(t-\tau)\, e^{-\sqrt{\lambda_i}\,\tau}\, d\tau \qquad j=1, 2, \ldots \tag{3.63}$$

$$b_j(0) = 0$$

Hence the complete solution to the self adjoint system specified by Eq. (3.46) is

$$z(x,t) = \sum_i \left[a_i(t) + b_i(t) \right]\, \phi_i(x) \tag{3.64}$$

where $a_i(t)$ and $b_i(t)$ are given by Eqs. (3.57) and (3.63).

Linear Formally Self Adjoint Damped Continuous Systems

The equation of motion of linear formally self adjoint damped continuous systems may be written as

$$p(x)\, u_{tt}(x,t) + L'_{2x}\, u_t(x,t) + L'_{1x}\, u(x,t) = f_1(x,t) \tag{3.65}$$

$$p(x) > 0, \qquad 0 \leq x \leq 1$$

where L'_{2x} and L'_{1x} are formally self adjoint operators.

The first term on the left hand side of Eq. (3.65) is the inertia term, the second is the generalized viscous damping term, while the third is the stiffness term. On the right hand side is the forcing function. As in the undamped case it is possible to rewrite Eq. (3.65) in canonical form as follows

$$z_{tt}(x,t) + L_{2x}\, z_t(x,t) + L_{1x}\, z(x,t) = f(x,t) \tag{3.66}$$

where

$$z(x,t) = \sqrt{p(x)}\; u(x,t)\;; \qquad f(x,t) = \frac{1}{\sqrt{p(x)}}\; f_1(x,t)$$

$$L_{2x} = \frac{1}{\sqrt{p(x)}}\; L'_{2x}\; \frac{1}{\sqrt{p(x)}}$$

$$L_{1x} = \frac{1}{\sqrt{p(x)}}\; L'_{1x}\; \frac{1}{\sqrt{p(x)}}$$

As L'_{1x} and L'_{2x} are formally self adjoint operators it is easy to see that so are L_{1x} and L_{2x}.

The boundary conditions on Eq. (3.65) must be equal in number to the maximum of the highest order of the derivatives in the operators L_{1x} and L_{2x}. However, there is no need for the boundary conditions to be such that L_{1x} and L_{2x} are self adjoint operators in the manifold spanned by functions which are acceptable solutions of the problem. It may happen, for example, that the highest order derivative in L_{2x} is lower than the highest order derivative in L_{1x}. In that case, only a subset of the boundary conditions are needed to define the domain of definition of L_{2x}. Needless to say, L_{2x} may not be self adjoint in this domain (although it is given that it is formally self adjoint) and

neither need L_{1x} be self adjoint in the domain defined by all the boundary conditions of the problem. The problems of interest in this work contain not only formally self adjoint operators but a special sub class contain self adjoint operators in the manifold of functions satisfying the boundary and interval conditions of the problem.

Linear Self Adjoint Damped Continuous Systems

The canonical form of the equations of motion of linear self adjoint damped continuous systems may be written as

$$z_{tt}(x, t) + L_{2x}\, z_t(x, t) + L_{1x}\, z(x, t) = f(x, t) \qquad (3.67)$$

where L_{2x} and L_{1x} are linear self adjoint operators in the manifold of functions satisfying the boundary conditions of the problem. The manifold of functions in which L_{1x} and L_{2x} are separately self adjoint may or may not have common functions. It should be noted that if one of the operators has a lower highest order derivative than the other, it is possible that the operator with the lower highest order derivative be self adjoint in a manifold of functions which excludes the manifold of functions satisfying all the boundary conditions of the problem. In other words, the boundary conditions of the entire problem may unnecessarily restrict the manifold of possible solutions as far as the self adjointness of the operator with the lower highest order derivative is concerned. It may happen that the boundary conditions do not restrict this manifold in any way, in which case, the boundary conditions of the problem are said to be compatible with the subset of

the boundary conditions required to specify the manifold in which the operator with the lower highest order derivative is self adjoint. This means, that in this case, any subset (sufficient in number) of the boundary conditions of the problem may be used to specify the domain of definition in which the operator with the lower highest order derivative is self adjoint and that this domain is independent of the choice of the subset of the boundary conditions. It should be noted, of course, that the domain of definition of L_{1x} and L_{2x} cannot be precisely the same in that the operator with the lower highest order derivative (say n) has a domain which includes functions which satisfy the boundary conditions and are continuously differential only (n-1) times (the n^{th} derivative being piece wise continuous).

To determine orthogonality conditions for the self adjoint damped continuous system specified by Eq. (3.67), a procedure analogous to Foss's method for discrete systems is used. To Eq. (3.67) add the identity

$$z_t(x, t) = y(x, t) \tag{3.68}$$

The homogeneous part of Eq. (3.67) may be rewritten using Eq. (3.68)

$$y_t(x, t) + L_{2x}\, z_t(x, t) + L_{1x}\, z(x, t) = 0 \tag{3.69}$$

Let

$$\begin{aligned} y_n(x, t) &= \beta_n(x)\, e^{\alpha_n t} \\ z_n(x, t) &= \phi_n(x)\, e^{\alpha_n t} \end{aligned} \tag{3.70}$$

where $\phi_n(x)$ and $\beta_n(x)$ satisfy the boundary conditions of the problem. Substituting Eq. (3.70) into Eqs. (3.68) and (3.69)

$$a_n \phi_n(x) = \beta_n(x) \tag{3.71}$$

$$a_n \beta_n(x) + a_n L_{2x} \phi_n(x) + L_{1x} \phi_n(x) = 0 \tag{3.72}$$

Multiply Eq. (3.71) by $\beta_m(x)$ and (3.72) by $\phi_m(x)$ and add the two equations and integrate over x from 0 to 1.

$$a_n \int_0^1 \left\{\phi_n(x)\ \beta_m(x) + \phi_m(x)\ \beta_n(x)\right\} dx + a_n \int_0^1 \phi_m(x)\ L_{2x}\ \phi_n(x)\ dx$$

$$+ \int_0^1 \phi_m(x)\ L_{1x}\ \phi_n(x) dx = \int_0^1 \beta_n(x)\ \beta_m(x)\ dx \tag{3.73}$$

In (3.74) interchanging subscripts n and m

$$a_m \int_0^1 \left\{\phi_m(x)\ \beta_n(x) + \phi_n(x)\ \beta_m(x)\right\} dx + a_m \int_0^1 \phi_n(x)\ L_{2x}\ \phi_m(x)\ dx$$

$$+ \int_0^1 \phi_n(x)\ L_{1x}\ \phi_m(x)\ dx = \int_0^1 \beta_n(x)\ \beta_m(x)\ dx \tag{3.74}$$

Subtracting Eq. (3.74) from Eq. (3.73) noting that L_{1x} and L_{2x} are self adjoint in the manifold spanned by functions $\phi_n(x)$ and $\beta_n(x)$

$$(a_n - a_m) \int_0^1 \left\{\phi_n(x)\ \beta_m(x) + \phi_m(x)\ \beta_n(x)\right\} dx + (a_n - a_m) \int_0^1 \phi_m(x) L_{2x} \phi_n(x) dx$$

$$= 0$$

$\therefore$ if $a_n \neq a_m$

$$\int_0^1 \left\{ \phi_n(x)\, \beta_m(x) + \phi_m(x)\, \beta_n(x) \right\} dx = -\int_0^1 \phi_n(x)\, L_{2x}\, \phi_m(x) dx \qquad (3.75)$$

$$n \neq m$$

Substituting Eq. (3.75) into Eq. (3.73)

$$\int_0^1 \phi_n(x)\, L_{1x}\, \phi_m(x)\, dx = \int_0^1 \beta_n(x)\, \beta_m(x)\, dx \qquad (3.76)$$

Equations (3.75) and (3.76) may be reduced, by using Eq. (3.71)

$$(a_n + a_m) \int_0^1 \phi_n(x)\, \phi_m(x) dx = -\int_0^1 \phi_n(x) L_{2x}\, \phi_m(x) dn \qquad (3.77)$$

$$n \neq m$$

$$\int_0^1 \phi_n(x) L_{1x}\, \phi_m(x) dx = a_n a_m \int_0^1 \phi_n(x)\, \phi_m(x) dx \qquad (3.78)$$

$$n \neq m$$

Equations (3.77) and (3.78) are the orthogonality conditions for the canonical self adjoint system specified by Eq. (3.67). By a suitable change of variables, it may be seen that the orthogonality relationships for the original system as specified by Eq. (3.65) are derived from Eqs. (3.77) and (3.78) by substituting

$$\sqrt{p(x)}\, \phi_n^*(x) = \phi_n(x) \qquad (3.79)$$

where

$$u_n(x, t) = \phi_n^*\, e^{a_n t} = \frac{1}{\sqrt{p(x)}}\, z_n(x, t)$$

to give

$$(a_n + a_m)\int_0^1 p(x)\,\phi_n^*(x)\,\phi_m^*(x)dx = \int_0^1 \phi_n^*(x)\,\sqrt{p(x)}\,L_{2x}\sqrt{p(x)}\,\phi^*(x)dx$$

$$= \int_0^1 \phi_n^*(x)\,L'_{2x}\,\phi^*(x)dx \qquad (3.80)$$

and

$$\int_0^1 \phi_n^*(x)\,\sqrt{p(x)}\,L_{1x}\sqrt{p(x)}\,\phi_m^*(x)dx = a_n a_m \int_0^1 p(x)\,\phi_n^*(x)\,\phi_m^*(x)dx$$

or

$$\int_0^1 \phi_n^*(x)\,L'_{1x}\,\phi_m^*(x)dx = a_n a_m \int_0^1 p(x)\,\phi_n^*(x)\,\phi_m^*(x)dx \qquad (3.81)$$

Equations (3.71) and (3.72) are in the form of an eigenvalue-eigenfunction equation and may be used to determine the sequence of eigenvalues a_n, n=1, 2, . . . and eigenfunctions ϕ_n, β_n, n=1, 2, ... The actual determination of the eigenfunctions and eigenvectors is far from trivial, and in general recourse must be had to numerical analysis. As in the discrete case the set of eigenfunctions need not be complete in which case it is necessary to determine the generalized eigenfunctions before completing the solution by mode methods. As is usual, generalized eigenfunctions are only associated with repeated eigenvalues.

Classical Normal Modes in Linear Self Adjoint Systems

In direct analogy to the discrete case, classical systems are

defined to be those self adjoint systems which possess a complete set of real orthonormal eigenfunctions, i. e., $\phi_n(x)$, $\phi_m(x)$ exists such that

$$\int_0^1 \phi_n(x)\, \phi_m(x)\, dx = 0 \qquad n \neq m \tag{3. 82}$$

Substituting Eq. (3. 82) into Eqs. (3. 77) and (3. 78) it is seen that in classical systems the following conditions also hold

$$\int_0^1 \phi_n(x)\, L_{1x}\, \phi_m(x) dx = 0 \qquad n \neq m \tag{3. 83}$$

$$\int_0^1 \phi_n(x)\, L_{2x}\, \phi_m(x) dx = 0 \qquad n \neq m \tag{3. 84}$$

As the set of eigenfunctions is complete, it is easy to see from Eqs. (3. 83) and (3. 84) that they must be the eigenfunctions of L_{1x} and L_{2x}. This in turn, requires that the boundary conditions be compatible, i. e., all the boundary conditions are derivable from the subset required for the self adjointness of the operator with the lower highest derivative. As an example, consider the following operators

$$L_{2x} = \frac{d^2}{dx^2}, \qquad L_{1x} = \frac{d^4}{dx^4} \tag{3. 85}$$

The operators defined by Eq. (3. 85) are formally self adjoint and L_{2x} will be self adjoint under the boundary condition

$$u(0) = u(1) = 0 \tag{3. 86}$$

However, the operator L_{1x} has a fourth order derivative and so requires four boundary conditions to determine its domain of

definition. One such set of boundary conditions could be

$$u(0) = u(1) = 0$$

$$\left.\frac{du}{dx}\right|_{x=0} = \left.\frac{du}{dx}\right|_{x=1} = 0 \tag{3.87}$$

L_{1x} is a self adjoint operator in the domain with boundary conditions specified by Eq. (3.87). But, these boundary conditions are not derivable from the subset used to define the domain in which L_{2x} is self adjoint. The boundary conditions of the domain in which L_{2x} is self adjoint are easily seen to be

$$u(0) = u(1) = 0$$

$$\left.\frac{d^2u}{dx^2}\right|_{x=0} = \left.\frac{d^2u}{dx^2}\right|_{x=1} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\left.\frac{d^{2m}u}{dx^{2m}}\right|_{x=0} = \left.\frac{d^{2m}u}{dx^{2m}}\right|_{x=1} = 0 \qquad m=0, 1, 2, \ldots \tag{3.88}$$

The first set of boundary conditions in Eq. (3.88) is the original boundary condition (Eq. 3.86) used to define a domain in which L_{1x} is self adjoint. The second and subsequent sets (m=1, 2, . . .) are derivable from the first by using the operator L_{1x} in the following manner. All the eigenfunctions u_i of L_{1x} satisfy equations of the following type

$$L_{1x}\, u_i = \lambda_i\, u_i \tag{3.89}$$

as

$$u_i(0) = u_i(1) = 0, \qquad L_{1x}\, u_i(x)\Big|_{x=0} = L_{1x}\, u_i(x)\Big|_{x=1} = 0$$

or

$$\left.\frac{d^2u_i}{dx^2}\right|_{x=0} = \left.\frac{d^2u_i}{dx^2}\right|_{x=1} = 0$$

This process may be continued, iteratively, to develop the sequence of boundary conditions specified by Eq. (3.88). As the set of eigenfunctions is complete and the boundary conditions are linear and homogeneous, a sequence of boundary conditions derived for the eigenfunctions are applicable to all functions in the domain of the operator.

Now, it can be seen from Eq. (3.58) that the boundary conditions used to specify the domain of L_{1x} (Eq. 3.87) actually exclude some of the functions in the domain of L_{2x}. In this particular case the eigenfunctions of L_{2x} under the original boundary conditions (Eq. 3.86) are

$$u_n(x) = \sin\ n\pi x \qquad n=1, 2, \ldots$$

Placing the restraints of the original boundary conditions of L_{1x} on the domain of L_{2x}, it is seen that such a domain is actually the null space.

Hence, the necessity of compatibility of boundary conditions has been demonstrated by this simple example. In a similar manner it can be shown that any two linear self adjoint differential operators can only possess the same complete set of eigenfunctions if their boundary conditions are compatible.

It is interesting to note that in classical systems, the eigenfunctions $\phi_i(x)$, $i=1, 2, \ldots$, associated with both real and complex eigenvalues are real. Whereas, the eigenfunction in Foss's formulation associated with real eigenvalues are always real, those

associated with complex eigenvalues need not be real. However, the reality of the eigenfunctions is not sufficient for classical systems in that they must be also orthogonal.

Necessary and Sufficient Conditions for Self-Adjoint Systems to be Classical

It will now be demonstrated that the necessary and sufficient conditions that the canonical form of the self adjoint continuous conditions system possesses classical normal modes is that

(i) L_{1x} and L_{2x} are self adjoint operators and the boundary conditions of the problem are such that the total set of boundary conditions are derivable from the subset required to specify the domain of definition of L_{1x} and L_{2x} (such boundary conditions are said to be compatible).

(ii) $$L_{1x}L_{2x} = L_{2x}L_{1x} \tag{3.90}$$

where the equivalence of the operators $L_{1x}L_{2x}$ and $L_{2x}L_{1x}$ is verified by expanding both sides of Eq. (3.90) and checking the coefficients of the various orders of the derivatives.

Proof of the sufficiency of the condition: Given that L_{1x} and L_{2x} are both self adjoint operators in the manifold of the functions satisfying the boundary and interval conditions of the problem (with suitable restrictions as to piecewise differentiability) it is required to be shown that the commutability of the operators (as given by Eq. (3.90)) is a sufficient condition for L_{1x} and L_{2x} to possess the same complete set of eigenfunctions. Consider

$$\langle \phi_m, L_{2x} \phi_n \rangle = I_o \tag{3.91}$$

where ϕ_m, m=1, 2, . . . , are the complete set of eigenfunctions associated with the undamped problem, i. e., with L_{1x} and the boundary conditions of the original problem. Here, it is tacitly assumed in Eq. (3.91) that the highest order derivative in L_{2x} is lower than the corresponding derivative in L_{1x}. If this is not so, the argument goes through if L_{1x} is interchanged for L_{2x}

$$\therefore \quad L_{1x} \phi_m = \lambda_m \phi_m \qquad m=1, 2, \ldots \tag{3.92}$$

On substituting Eq. (3.92) into Eq. (3.91)

$$I_o = \frac{1}{\lambda_n} \langle \phi_m, L_{2x} L_{1x} \phi_n \rangle \tag{3.93}$$

Utilizing the commutability condition Eq. (3.93) may be rewritten as

$$I_o = \frac{1}{\lambda_n} \langle \phi_m, L_{1x} L_{2x} \phi_n \rangle \tag{3.94}$$

As L_{1x} is a self adjoint operator and the boundary conditions are compatible, Eq. (3.94) may be rewritten.

$$I_o = \frac{1}{\lambda_n} \langle L_{2x} \phi_n, L_{1x} \phi_m \rangle = \frac{\lambda_m}{\lambda_n} \langle \phi_m, L_{2x} \phi_n \rangle \tag{3.95}$$

Hence, from Eqs. (3.91) and (3.95) if $\lambda_n \neq \lambda_m$

$$\langle \phi_m, L_{2x} \phi_n \rangle = 0 \qquad n \neq m \; . \tag{3.96}$$

This completes the proof of the sufficiency of the classically damped conditions (Eq. (3.90)) for systems whose undamped eigenvalues are distinct. The proof of the sufficiency of the conditions in the case of

equal roots will be taken up later.

Proof of the necessity of the conditions given by Eq. (3.90) for systems to be classical: Given that L_{1x} and L_{2x} are self adjoint operators possessing a common complete set of eigenfunctions, i.e., the system is classical

$$\int_0^1 \phi_m(x)\, L_{1x}\, \phi_n(x)dx = 0 \qquad n \neq m$$

$$\int_0^1 \phi_m(x)\, L_{2x}\, \phi_n(x)dx = 0 \qquad n \neq m$$

$$\int_0^1 \phi_m(x)\, \phi_n(x)dx = 0 \qquad n \neq m$$

$$\int_0^1 \phi_m(x)\, \phi_n(x)dx = 1 \qquad \text{all } n \qquad n, m=1, 2, \ldots \qquad (3.97)$$

It is required to be shown that the conditions listed in Eq. (3.90) are necessary.

If the operators L_{1x} and L_{2x} commute and the boundary conditions are compatible with the self adjointness of L_{1x} and L_{2x} it will now be shown that $L_{1x}L_{2x}$ and $L_{2x}L_{1x}$ are both self adjoint operators in the domain specified by the boundary and interval conditions of the problem. Consider

$$I_1 = \langle u, L_{1x}L_{2x}v \rangle \qquad (3.98)$$

where u and v are functions which satisfy the linear homogeneous boundary conditions of the problem.

If $L_{1x}L_{2x} = L_{2x}L_{1x}$

$$I_1 = \langle u, L_{2x} L_{1x} v \rangle \tag{3.99}$$

As L_{2x} is a self adjoint operator and as the boundary conditions are compatible it may be seen that Eq. (3.99) may be rewritten as

$$\begin{aligned} I_1 &= \langle L_{1x} v,\ L_{2x} u \rangle \\ &= \langle L_{2x} u,\ L_{1x} v \rangle \end{aligned} \tag{3.100}$$

As L_{1x} is a self adjoint operator and the boundary conditions are compatible, Eq. (3.100) may be rewritten as

$$I_1 = \langle v,\ L_{1x} L_{2x}\, u \rangle \tag{3.101}$$

From Eqs. (3.98) and (3.101) it is seen that $L_{1x} L_{2x}$ and similarly $L_{2x} L_{1x}$, is a self adjoint operator in the coinciding domains of definition of L_{1x} and L_{2x}. The converse may be established in a similar manner. To continue the proof of the necessity of the conditions given by Eq. (3.90) for systems to be classical, it is seen from Eq. (3.97), since the eigenfunctions form a complete set

$$\begin{aligned} L_{1x}\, \phi_n(x) &= c_n\, \phi_n(x) \\ L_{2x}\, \phi_n(x) &= d_n\, \phi_n(x) \qquad\qquad n=0, 1, \ldots \end{aligned} \tag{3.102}$$

Any two functions u(x) and v(x) satisfying the boundary conditions may be expanded in an infinite series of eigenfunctions as follows

$$u(x) = \sum_m a_m \phi_m(x)\ ;\quad v(x) = \sum_n b_n \phi_n(x) \tag{3.103}$$

Consider

$$I_2 = \langle v, L_{2x} L_{1x}\, u \rangle \tag{3.104}$$

On substituting Eq. (3.103) into Eq. (3.104)

$$I_2 = \sum_n \sum_m a_n b_m c_n d_m \langle \phi_n(x), \phi_m(x) \rangle$$

Likewise, let

$$I_3 = \langle u, L_{2x} L_{1x} v \rangle = \sum_n \sum_m a_n b_m c_n d_m \langle \phi_n(x) \phi_m(x) \rangle \qquad (3.105)$$

$$\therefore \; I_2 = I_3$$

Therefore $L_{2x}L_{1x}$ is a self adjoint operator in the domain specified by the boundary and interval conditions of the problem. Hence, as L_{1x} and L_{2x} are separately self adjoint in the same domain

$$L_{1x}L_{2x} = L_{2x}L_{1x}$$

and all the conditions specified by Eq. (3.90) are satisfied. The proof of the sufficiency of the conditions in the case of equal roots will now be taken up. A somewhat more general proof than that given previously for the case of distinct eigenvalues is presented in which the arguments are based on the self adjointness of the operator $L_{2x}L_{1x}$ rather than on the commutability of the operators $L_{1x}L_{2x}$. Given that L_{1x}, L_{2x} and $L_{1x}L_{2x}$ are self adjoint operators in the domain specified by the boundary and interval conditions of the problem, it is required to be shown that even in the case in which L_{1x} or L_{2x} possesses eigenvalues of multiplicity greater than 1, the system is classically damped.

Consider

$$\langle u, L_{2x} L_{1x} v \rangle = \langle v, L_{2x} L_{1x} u \rangle \qquad (3.106)$$

where u and v satisfy the boundary conditions of the problem. Let

u_i, i=1, 2, . . . be the eigenfunctions of L_{1x} and

λ_i, i=1, 2, . . . the corresponding eigenvalues.

As the u_i, i=1, 2, . . . , form a complete set u and v may be represented in the following series

$$u = \sum_n a_n u_n \quad ; \quad v = \sum_n b_n u_n \tag{3.107}$$

On substituting Eq. (3.107) into (3.106)

$$\left\langle \sum_n a_n u_n, L_{2x} \sum_n \lambda_n b_n u_n \right\rangle = \left\langle \sum b_n u_n, L_{2x} \sum_n \lambda_n a_n u_n \right\rangle \tag{3.108}$$

Let

$$L_{2x} u_n = \sum_m c_{nm} u_m \tag{3.109}$$

As u_n, n=1, 2, . . . , the eigenfunctions of L_{1x} satisfy the boundary conditions which are compatible with the self adjointness of L_{1x} and L_{2x}, they are contained in the domain of definition of L_{2x}. Hence,

$$\langle u_m, L_{2x} u_n \rangle = \langle u_n, L_{2x} u_m \rangle \tag{3.110}$$

On substituting Eq. (3.109) into Eq. (3.110) and using the orthonormalization conditions of the eigenfunctions of a self adjoint operator

$$c_{nm} = c_{mn} \tag{3.111}$$

On substituting Eq. (3.109) into Eq. (3.108) and using the orthonormalization conditions of the u_n's

$$\sum_m \sum_n a_m b_n \lambda_n c_{nm} = \sum_m \sum_n a_m b_n \lambda_m c_{mn} \tag{3.112}$$

On substituting Eq. (3.111) into Eq. (3.112)

$$\sum_m \sum_n a_m b_n c_{nm} (\lambda_n - \lambda_m) = 0 \tag{3.113}$$

As a_m, b_n $m, n=1, 2, \ldots$ are arbitrary,

$$c_{nm} = 0 \qquad \lambda_n \neq \lambda_m \qquad n, m, =1, 2, \ldots \tag{3.114}$$

c_{nn} need not be zero

From Eq. (3.109) these observations ($\lambda_n \neq \lambda_m$) lead directly to the results already derived above.

If, however, $\lambda_n = \lambda_m$, $n, m, =1, 2, \ldots$, some further work is required before reaching the desired conclusion. Suppose there is an eigenvalue λ_1, of L_{1x} of multiplicity a, i.e., there are a eigenfunctions associated with this eigenvalue. For convenience, let these eigenfunctions be

$$u_1, u_2, u_3 \text{ -----} u_a$$

where these functions are orthonormalized (using the Gram-Schmidt process if necessary).

Forming from this set of eigenfunctions another set of orthonormal eigenfunctions as follows

$$u_i^* = \sum_{j=1}^{a} g_{ij} u_j \qquad i=1, 2, \ldots a \tag{3.115}$$

it is seen that the orthonormalization condition requires that

$$\left[g_{ij}\right]^T = \left[g_{ij}\right]^{-1} \tag{3.116}$$

The sufficiency of the conditions for systems to be classical, (i.e., $L_{2x}L_{1x}$ be a self adjoint operator in the domain in which L_{1x} and L_{2x} are separately self adjoint) in the case in which L_{1x} has eigenvalues of multiplicity greater than 1, follows once it is shown that it is possible to select a set of g_{ij}'s $i, j=1, 2, \ldots a$, such that

$$L_{2x}u_i^* = c_{ii}^* u_i^* \tag{3.117}$$

Now, from Eqs. (3.109) and (3.114)

$$L_{2x}u_i = \sum_{n=1}^{a} c_{ij} u_j \qquad i=1, 2, \ldots a \tag{3.118}$$

and from Eqs. (3.115) and (3.118)

$$L_{2x}u_i^* = \sum_{\ell=1}^{a} \sum_{j=1}^{a} g_{i\ell} c_{\ell j} u_j \qquad i=1, 2, \ldots a \tag{3.119}$$

$\therefore$ from Eqs. (3.117) and (3.119) if

$$c_{ii}^* u_i^* = \sum_{\ell=1}^{a} \sum_{j=1}^{a} g_{ij} c_{\ell j} u_j \tag{3.120}$$

the sufficiency of the condition has been demonstrated. Substituting Eq. (3.115) into Eq. (3.120)

$$\sum_{j=1}^{a} c_{ii}^* g_{ij} u_j = \sum_{j=1}^{a} \sum_{\ell=1}^{a} g_{i\ell} c_{\ell j} u_j \tag{3.121}$$

or as u_j, $j=1, 2, \ldots,$ are linearly independent

$$\sum_{\ell=1}^{a} g_{i\ell}\, c_{\ell j} = c^{*}_{ii}\, g_{ij} \tag{3.122}$$

which may be conveniently written as

$$\left[g_{ij}\right]\left[c_{ij}\right] = \left[c^{*}_{ii}\right]\left[g_{ij}\right] \tag{3.123}$$

where $\left[c_{ij}\right]$ is a $a \times a$ symmetric matrix whose ij^{th} element is c_{ij}, $\left[c^{*}_{ii}\right]$ is a diagonal $a \times a$ matrix whose ii^{th} element is c^{*}_{ii}. From Eqs. (3.116) and (3.123)

$$\left[g_{ij}\right]\left[c_{ij}\right]\left[g_{ij}\right]^{T} = \left[c^{*}_{ii}\right] \tag{3.124}$$

As $\left[c^{*}_{ii}\right]$ is a diagonal matrix, and $\left[c_{ij}\right]$ is a symmetric matrix, $\left[g_{ij}\right]$ exists such that Eq. (3.124) is satisfied. This concludes the proof of the sufficiency of the conditions specified by Eq. (3.90) for systems to be classical if the eigenvalues of L_{1x} are repeated.

Therefore the conditions outlined by Eq. (3.90) are both necessary and sufficient conditions for self adjoint systems to be classical.

Example of a Classically Damped Continuous System

Consider the two operators

$$L_{1x} = \frac{d^2}{dx^2}\, a(x)\, \frac{d^2}{dx^2} \quad ; \quad L_{2x} = \frac{d}{dx}\, b(x)\, \frac{d}{dx} \tag{3.125}$$

L_{1x} and L_{2x} are formally self adjoint operators. It is easy to show that L_{2x} is a self adjoint operator in the manifold spanned by functions with boundary conditions

$$u(0) = u(1) = 0 \qquad 0 \le x \le 1 \tag{3.126}$$

To obtain compatible boundary conditions for the problem, consider the eigenvalue-eigenfunction equation

$$L_{2x}u_i = b'(x)\frac{du_i}{dx} + b(x)\frac{d^2u_i}{dx^2} = \lambda u_i \tag{3.127}$$

Using the boundary conditions specified by Eq. (3.126) the following set of compatible boundary conditions are derived

$$\begin{aligned} B_1(u):\ & b'(0)\frac{du}{dx}\bigg|_{x=0} + b(0)\frac{d^2u}{dx^2}\bigg|_{x=0} = 0 \\ B_2(u):\ & b'(1)\frac{du}{dx}\bigg|_{x=1} + b(1)\frac{d^2u}{dx^2}\bigg|_{x=1} = 0 \end{aligned} \tag{3.128}$$

Now $L_{1x}L_{2x} = L_{2x}L_{1x}$ if

$$\frac{d^2}{dx^2}\left[a(x)\frac{d^3}{dx^3}\left\{b(x)\frac{d}{dx}\right\}\right] = \frac{d}{dx}\left[b(x)\frac{d^3}{dx^3}\left\{a(x)\frac{d^2}{dx^2}\right\}\right] \tag{3.129}$$

On expanding Eq. (3.129) and equating the coefficients of the various orders of derivatives to zero, the following conditions must be satisfied

$$a(x) = b^2(x)$$

$$b^2(x)\frac{d^3b(x)}{dx^3} = 0 \quad \text{or if } b(x) \neq 0,\ \frac{d^3b(x)}{dx^3} = 0 \tag{3.130}$$

for the commutability of the operators. These results show how restrictive is the class of classical systems.

Constructive Sufficient Conditions for Systems to be Classical

In an analogous fashion to the Caughey series in the discrete case, it will now be shown that if, in the canonical form of the system, L_{2x} may be expanded in a power series

$$L_{2x} = \sum_{n=0}^{N} a_n (L_{1x})^n \tag{3.131}$$

the system possesses classical normal modes. The a_n's are so chosen that the eigenvalues of L_{2x} are finite. Consider a typical term of the series given by Eq. (3.131)

$$a_n (L_{1x})^n = a_n \underbrace{L_{1x} L_{1x} \cdots L_{1x}}_{\text{n terms}} \tag{3.132}$$

As L_{1x} is a self adjoint operator, it possesses a complete set of eigenfunctions $\phi_i(x)$, i=1, 2, ... It will now be shown that if L_{2x} is given by (3.131) these eigenfunctions are also eigenfunctions of L_{2x}. From Eq. (3.132)

$$a_n (L_{1x})^n \phi_i(x) = a_n (\lambda_i)^n \phi_i(x) \tag{3.133}$$

From Eq. (3.131)

$$L_{2x} \phi_i(x) = \sum_{n=0}^{N} a_n (\lambda_i)^n \phi_i(x) = a_i \phi_i(x) \tag{3.134}$$

$$\therefore \langle \phi_j, L_{2x} \phi_i(x) \rangle = 0 \qquad i \neq j. \tag{3.135}$$

From Eqs. (3.134) and (3.135) it may be seen that the system possesses classical normal modes if L_{1x} is a self adjoint operator, the boundary conditions are compatible and L_{2x} can be expanded as in Eq. (3.131).

In the discrete case it was possible to show that in the case of distinct eigenvalues the Caughey series is both a necessary and sufficient condition for classical systems. If a proof of the necessity of the condition given by Eq. (3.131) is attempted in the continuous case,

a major mathematical difficulty, by way of an infinite dimensional Vandernonde determinant, arises. Unfortunately, the literature is singularly scarce in references to such determinants. However, it does appear intuitively reasonable that Eq. (3.131) is both a necessary and sufficient condition for systems to be classical in the case of distinct eigenvalues of $L_{1x}(N \to \infty)$

Solution of Classical and Non Classical Self Adjoint Systems

The canonical form of the equation of motion of damped continuous systems may be written as

$$u_{tt}(x,t) + L_{2x}\, u_t(x,t) + L_{1x}\, u(x,t) = F(x,t) \tag{3.136}$$

If the system is classical, the solution goes through in exactly the same manner as the solution of the undamped problem treated above. Here attention will be confined to the non classical self adjoint system.

To Eq. (3.136) add the identity

$$u_t(x,t) = z(x,t) \tag{3.137}$$

Assuming that all the eigenfunctions of Eq. (3.136) are ordinary eigenfunctions, i. e., there are no generalized eigenfunctions, let

$$\begin{aligned} u(x,t) &= \sum_n \phi_n(x)\, \xi_n(t) \\ z(x,t) &= \sum_n \beta_n(x) \dot{\xi}_n(t) \end{aligned} \tag{3.138}$$

Substituting Eq. (3.138) into Eqs (3.136) and (3.137)

$$i)\quad \sum_n \beta_n(x)\,\dot{\xi}_n(t) + \sum_n L_{2x}\,\phi_n(x)\,\dot{\xi}_n(t) + \sum_n L_{1x}\,\phi_n(x)\,\xi_n(t) = F(x,t)$$

$$ii)\quad \sum_n \phi_n(x)\,\dot{\xi}_n(t) = \sum_n \beta_n(x)\,\xi_n(t) \tag{3.139}$$

Multiply both sides of Eq. i) (3.139) by $\phi_m(x)$ and Eq. ii) (3.139) by $\beta_m(x)$ add the resulting equations and integrate with respect to x from 0 to 1.

$$\sum_n \int_0^1 \left\{ \dot{\xi}_n(t) \left[\beta_n(x)\,\phi_m(x) + \phi_m(x) L_{2x} \phi_n(x) + \beta_m(x)\,\phi_n(x) \right] \right\} dx$$

$$+ \sum_n \int_0^1 \xi_n(t) \left[\phi_m(x) L_{1x} \phi_n(x) - \beta_n(x)\beta_m(x) \right] dx = \int_0^1 F(x,t)\,\phi_m(x)\,dx \tag{3.140}$$

Using the orthogonality conditions derived previously (Eqs. (3.77) and (3.78)), Eq. (3.140) may be reduced to

$$M_m\,\dot{\xi}_m(t) + K_m\,\xi_m(t) = F_m(t) \tag{3.141}$$

$$M_m = \int_0^1 \left[2\beta_m(x)\,\phi_m(x) + \phi_m(x) L_{2x}\,\phi_m(x) \right] dx$$

$$K_m = \int_0^1 \left[\phi_m(x) L_{1x}\,\phi_m(x) - \beta_m{}^2(x) \right] dx$$

$$F_m(t) = \int_0^1 F(x,t)\,\phi_m(x)\,dx \tag{3.142}$$

To reduce Eq. (3.141) still further, consider the homogeneous problem

$$u_{tt}(x,t) + L_{2x}\, u_t(x,t) + L_{1x}\, u(x,t) = 0 \tag{3.143}$$

Let

$$u(x,t) = \phi_m(x) e^{\alpha_m t}$$

$$\therefore\ \alpha_m^2\, \phi_m(x) + \alpha_m L_{2x}\, \phi_m(x) + L_{1x}\, \phi_m(x) = 0 \tag{3.144}$$

Multiply both sides of Eq. (3.144) by $\phi_m(x)$ and integrate with respect to x between 0 and 1.

$$\alpha_m^2 \int_0^1 \phi_m^2(x)\,dx + \alpha_m \int_0^1 \phi_m(x) L_{2x} \phi_m(x)\,dx + \int_0^1 \phi_m(x) L_{1x}\, \phi_m(x)\,dx = 0 \tag{3.145}$$

Using the definitions given by Eqs. (3.138) and (3.142), Eq. (3.145) may be reduced to

$$\alpha_m M_m + K_m = 0 \tag{3.146}$$

Therefore, using Eq. (3.146), Eq. (3.140) may be simplified to

$$\dot{\xi}_m(t) - \alpha_m\, \xi_m(t) = \frac{1}{M_m}\, F_m(t) \qquad M_m \neq 0 \tag{3.147}$$

$$\therefore\ \xi_m(t) = A_m e^{\alpha_m t} + \int_0^t \frac{1}{M_m}\, F_m(\tau)\, e^{\alpha_m (t-\tau)}\, d\tau \qquad m=1, 2, \ldots \tag{3.148}$$

Hence

$$u(x,t) = \sum_{m=1}^{\infty} \left[A_m e^{\alpha_n t} + \int_0^t \frac{1}{M_m}\, F_m(\tau)\, e^{\alpha_m (t-\tau)}\, d\tau \right] \phi_m(x) \tag{3.149}$$

where A_m, m=1, 2, . . . , are arbitrary constants.

The constants A_m, m=1, 2, . . . , are determined in the following manner. Suppose

$$u(x, 0) = u_o(x)$$

$$u_t(x, 0) = u_{ot}(x)$$

From Eq. (3.149)

$$u_o(x) = \sum_m A_m \phi_m(x) \tag{3.150}$$

$$u_{ot}(x) = \sum_m A_m a_m \phi_m(x) \tag{3.151}$$

Multiplying both sides of Eq. (3.150) by $\beta_n(x) + L_{2x} \phi_n(x)$ and both sides of Eq. (3.151) by $\phi_n(x)$, add the resulting equations

$$u_o(x)\beta_n(x) + u_o(x)L_{2x}\phi_n(x) + u_{ot}(x)\phi_n(x)$$

$$= \sum_m A_m \left\{\phi_m(x)\beta_n(x) + \phi_m(x)L_{2x}\phi_n(x) + a_m\phi_m(x)\phi_n(x)\right\} \tag{3.152}$$

Integrate both sides of Eq. (3.152) with respect to x from 0 to 1.

$$\int_0^1 \left\{u_o(x)\beta_n(x) + u_o(x)L_{2x}\phi_n(x) + u_{ot}(x)\phi_n(x)\right\} dx$$

$$= \sum_m A_m \int_0^1 \left\{\phi_m(x)\beta_n(x) + \phi_m(x)L_{2x}\phi_n(x) + a_m\phi_m(x)\phi_n(x)\right\} dx \tag{3.153}$$

Using the orthogonality condition Eq. (3.77) and the definition of M_n given by Eq. (3.142)

$$A_n = \frac{1}{M_n} \int_0^1 \left\{ u_o(x)\beta_n(x) + u_o(x)L_{2x}\,\phi_n(x) + u_{ot}(x)\,\phi_n(x) \right\} dx$$

$$n=1, 2, \ldots$$

This concludes the treatment of the differential formulation of the continuous system. As noted above, there is a direct analogy between the solutions of the discrete and continuous systems. All the remarks concerning generalized eigenvectors apply equally well to generalized eigenfunctions but the practical difficulty of determining generalized eigenfunctions hardly warrants any extended discussion of systems possessing generalized eigenfunctions.

CHAPTER 4

FURTHER RESULTS FOR CONTINUOUS SYSTEMS

Introduction

In this chapter, the discussion of continuous systems is continued. The differential formulation of the undamped system of the last chapter is shown to be equivalent to an integral formulation. Self adjointness in the differential formulation corresponds to symmetry of the kernel in the integral formulation. A brief discussion is given on the analytical solution of integral equations. The concept of viscous damping as an integral operator is introduced. Necessary and sufficient conditions for damped integral systems to be classical are derived.

Finally, the solution of continuous systems by numerical methods is discussed. Some results on error bounds are noted.

Theory

The Equivalence of the Differential and Integral Formulations of the Equations of Motion of Undamped Continuous Systems

Given a self adjoint operator L_x, defined over a finite interval $(0 \rightarrow 1)$ with suitable boundary conditions $B_1(u) = B_2(u) = 0$. Further, assume that $\lambda = 0$ is not an eigenvalue of the operator, i.e.,

$$L_x u = 0 \qquad 0 \leq x \leq 1 \tag{4.1}$$

has only the trivial solution $u = 0$.

Now consider the function y(x, t) such that

$$L_x\, y(x, \eta) = 0 \qquad x \neq \eta$$

$$B_1(y(x,\eta)) = 0 \;\; ; \;\; B_2(y(x,\eta)) = 0 \tag{4.2}$$

where $y(x, \eta)$ satisfies Eq. (4.1) at every point in the interval $0 \leq x \leq 1$ but the point $x = \eta$. At this point (which in an arbitrarily selected point η in the interval) $y(x, \eta)$ must be continuous but there are no demands as to the existence of the derivatives of $y(x, \eta)$ at $x = \eta$. This means in effect that $y(x, \eta)$ is not a function in the domain of definition of L_x (e. g., if the highest order derivative in L_x is two then all functions in the domain of definition of L_x must have continuous first derivatives). To determine $y(x, \eta)$ unambiguously,[42] it is necessary to impose certain conditions regarding the permissible discontinuities of its (n-1)-th derivative where n is the highest order derivative in L_x. The actual determination of $y(x, \eta)$ is not difficult if the following step by step technique is adopted.

In line with the discussion above $y(x, \eta)$ satisfies the following equation

$$L_x\, y(x, \eta) = \delta(x-\eta) \qquad 0 \leq x \leq 1 \tag{4.3}$$

where $\delta(x-\eta)$ is Dirac's delta function, i. e.,

$$\int_{\eta_-}^{\eta_+} \delta(x-\eta)\,dx = 1 \quad ; \quad 0 \leq \eta \leq 1$$

Formally, integrating both sides of Eq. (4.3) with respect to x from η_- to η_+

$$\int_{\eta_-}^{\eta_+} L_x y(x, \eta)\,dx = 1 \tag{4.4}$$

Equation (4. 4) is the jump condition at $x = \eta$ for the $(n-1)^{th}$ derivative of $y(x, \eta)$ where n is the highest order derivative in L_x. All other derivatives must be at least piecewise continuous. The technique to use to determine $y(x, \eta)$ is as follows:

Divide $y(x, \eta)$ in two ranges

$$y(x, \eta) = \begin{cases} y_\ell(x, \eta) & 0 \le x \le \eta \\ y_r(x, \eta) & \eta \le x \le 1 \end{cases}$$

As the highest order derivative in L_x is n there are n linearly independent solutions $(u=1, x, x^2 \cdots x^{n-1})$ of the following equation

$$L_x u = 0 \tag{4.5}$$

neglecting any considerations as to boundary conditions. Consider

$$\begin{aligned} y_\ell(x, \eta) &= a_{\ell 0} + a_{\ell 1}x + a_{\ell 2}x^2 + \cdots + a_{\ell n-1}x^{n-1} \\ y_r(x, \eta) &= a_{r 0} + a_{r 1}x + a_{r 2}x^2 + \cdots + a_{r n-1}x^{n-1} \end{aligned} \tag{4.6}$$

Now $y(x, \eta)$ must satisfy the boundary conditions

$$\begin{aligned} B_1(y(x, \eta)) &= 0 \\ B_2(y(x, \eta)) &= 0 \\ &\vdots \\ B_n(y(x, \eta)) &= 0 \end{aligned} \tag{4.7}$$

Substitution of the expressions for $y_\ell(x, \eta)$ and $y_r(x, \eta)$ from Eq. (4. 6) into the boundary conditions specified by Eq. (4. 7) (which in

general will involve derivatives of $y_\ell(x, \eta)$ and $y_r(x, \eta)$ at the end points $x = 0$, $x = 1$) leads to n linear equations in the 2n unknowns $a_{\ell o}, a_{\ell 1}, \cdots a_{\ell n-1}, a_{ro}, a_{r1}, \cdots a_{rn-1}$. A further (n-1) linear equations are derived from the following sequence of equations

$$\begin{aligned} y_\ell(x, \eta)\big|_{x=\eta} &= y_r(x, \eta)\big|_{x=\eta} \\ y_\ell^1(x, \eta)\big|_{x=\eta} &= y_r^1(x, \eta)\big|_{x=\eta} \\ &\vdots \qquad\qquad \vdots \\ y_\ell^{n-2}(x, \eta)\big|_{x=\eta} &= y_r^{n-2}(x, \eta)\big|_{x=\eta} \end{aligned} \tag{4.8}$$

where

$$y_\ell^j(x, \eta) = \frac{d^j}{dx^j}\, y_\ell(x, \eta)$$

This set of equations arise from the specification of the piecewise continuity of the first (n-2) derivatives of $y(x, \eta)$. The final equation for the determination of the 2n unknowns in Eq. (4.6) is given by the jump condition for the $(n-1)^{th}$ derivative of $y(x, \eta)\{x=\eta$, Eq. (4.4)$\}$.

A simple example will clarify the details of this step by step technique for determining $y(x, \eta)$.

Consider

$$L_x = \frac{d^2}{dx^2}\, a(x)\, \frac{d^2}{dx^2} \tag{4.9}$$

It is easy to show that L_x as given by Eq. (4.9) is a self adjoint operator in the following manifold

$$\begin{aligned} &u(0) = u(1) = 0 \\ &u'(0) = u'(1) = 0 \qquad 0 \leq x \leq 1 \end{aligned} \tag{4.10}$$

The jump condition may be determined from Eq. (4. 4)

$$\int_{\eta_-}^{\eta_+} \frac{d^2}{dx^2}\, a(x) \frac{d^2}{dx^2}\, y(x, \eta)\, dx = 1$$

$$\therefore \quad \int_{\eta_-}^{\eta_+} d\left(\frac{d}{dx}\, a(x) \frac{d^2}{dx^2}\, y(x, \eta) \right) = 1$$

$$\frac{d}{dx}\, a(x) \frac{d^2}{dx^2}\, y(x, \eta) \Big|_{\eta_-}^{\eta_+} = 1 \qquad (4.11)$$

Equation (4. 11) will be satisfied by the following conditions

$$\text{(i)} \qquad a(x) \frac{d^3}{dx^3}\, y(x, \eta) \Big|_{x=\eta_-}^{x=\eta_+} = 1 \qquad (4.12)$$

$$\text{(ii)} \qquad \frac{d^2}{dx^2}\, y(x, \eta) \qquad \text{is a piecewise continuous at } x = \eta$$

(i) may be further reduced to the jump condition for the third derivative of $y(x, \eta)$

$$\frac{d^3}{dx^3}\, y(x, \eta) \Big|_{x=\eta_-}^{x=\eta_+} = \frac{1}{a(\eta)} \qquad a(\eta) \neq 0 \qquad (4.13)$$

In line with Eq. (4. 6), let

$$y_\ell(x, \eta) = a_{\ell o} + a_{\ell 1} x + a_{\ell 2} x^2 + a_{\ell 3} x^3$$

$$y_r(x, \eta) = a_{ro} + a_{r1} x + a_{r2} x^2 + a_{r3} x^3$$

On substituting the boundary conditions (Eq. (4. 10))

$$a_{\ell o} = a_{ro} + a_{r1} + a_{r2} + a_{r3} = 0$$

$$a_{\ell 1} = a_{r1} + 2a_{r2} + 3a_{r3} = 0 \qquad (4.14)$$

Using Eq. (4.8) (n=4) the following sequence of equations may be developed.

Continuity of $y'(x, \eta)$ at $x = \eta$

$$a_{\ell 2}\eta^2 + a_{\ell 3}\eta^3 = a_{ro} + a_{r1}\eta + a_{r2}\eta^2 + a_{r3}\eta^3 \qquad (4.15)$$

Continuity of $y'(x, \eta)$ at $x = \varkappa$

$$2a_{\ell 2}\eta + 3a_{\ell 3}\eta^2 = a_{r1} + 2a_{r2}\eta + 3a_{r3}\eta^2 \qquad (4.16)$$

Continuity of $y''(x, \eta)$ at $x = \eta$

$$2a_{\ell 2} + 6a_{\ell 3}\eta = 2a_{r2} + 6a_{r3}\eta \qquad (4.17)$$

Jump Condition of $y'''(x, \eta)$ at $x = \eta$ (Eq. 4.13)

$$-6a_{\ell 3} + 6a_{r3} = \frac{1}{a(\eta)} \qquad (4.18)$$

From Eqs. (4.14), (4.15), (4.16), (4.17) and (4.18) it may be seen that

$$a_{\ell o} = a_{\ell 1} = 0 \; ; \quad a_{\ell 2} = \frac{\eta}{2a(\eta)} - \frac{\eta^2}{a(\eta)} + \frac{\eta^3}{2a(\eta)}$$

$$a_{ro} = -\frac{\eta^3}{6a(\eta)} \; ; \quad a_{r1} = \frac{\eta^2}{2a(\eta)} \; ; \quad a_{r2} = -\frac{\eta^2}{a(\eta)} + \frac{1}{2}\frac{\eta^3}{a(\eta)}$$

$$a_{\ell 3} = -\frac{1}{6a(\eta)} + \frac{\eta^2}{2a(\eta)} - \frac{\eta^3}{3a(\eta)} \; ; \quad a_{r3} = \frac{\eta^2}{2a(\eta)} - \frac{\eta^3}{3a(\eta)}$$

Hence, $y(x, \eta)$ has been determined uniquely.

One of the most useful characteristics of the Green's function, $y(x, \eta)$, is that it is a symmetric function in x and η provided the

corresponding operator L_x is a self adjoint operator in a given domain of definition. It is this property of the Green's function which gives the formulation of the self adjoint continuous system as an integral equation some advantages over the differential formulation.

From Eq. (4. 3) it may be seen that

$$\langle y(x, \xi), L_x u(x) \rangle = \langle u(x), L_x y(x, \xi) \rangle = u(\xi)$$

since $y(x, \xi)$ satisfies the boundary and interval condition of the domain of definition of L_x.

The Undamped Continuous System Formulated as an Integral Equation

Consider the inhomogeneous undamped vibrating system

$$m(x)\, z_{tt}(x, t) + L_{1x}\, z(x, t) = f(x; t) \tag{4. 19}$$

Let $y_1(x, \eta)$ be the Green's function associated with the linear differential self adjoint operator L_{1x} (satisfying the appropriate boundary conditions of the problem).

Multiply both sides of Eq. (4. 19) by $y_1(\eta, x)$ and integrate each term with respect to x from 0 to 1.

$$\int_0^1 m(x)\, y_1(\eta, x)\, z_{tt}(x, t)dx + z(\eta, t) = \int_0^1 y_1(\eta, x) f(x, t)dx \tag{4. 20}$$

To reduce Eq. (4. 20) to canonical form let

$$u(\eta, t) = \left| m(\eta) \right|^{1/2} z(\eta, t)$$

$$\therefore \int_0^1 \left|m(\eta)\right|^{1/2} y_1(\eta, x) \left|m(x)\right|^{1/2} u_{tt}(\eta, t)dx + u(\eta, t) = F(\eta, t) \qquad (4.21)$$

or

$$\int_0^1 y(\eta, x)\, u(x, t)_{tt}\, dx + u(\eta, t) = F(\eta, t) \qquad (4.22)$$

where

$$y(\eta, x) = \left|m(\eta)\right|^{1/2} y_1(\eta, x) \left|m(x)\right|^{1/2} = y(x, \eta)$$

$$F(x, t) = \int_0^1 \left|m(\eta)\right|^{1/2} y_1(\eta, x)\, f(x, t)dx$$

If

$$F(x, t) = F_1(x)\, e^{\sqrt{a}\, t} \qquad (4.23)$$

Eq. (4.22) may be solved by substituting

$$u(\eta, t) = u_1(\eta)\, e^{\sqrt{a}\, t}$$

$$\therefore\ a \int_0^1 y(\eta, x)\, u_1(x)dx + u_1(\eta) = F_1(\eta) \qquad (4.24)$$

If $F(x, t)$ cannot be expanded in a series of terms of type (4.23) (to give a series of linear integral equations of type (4.24)) the Fourier transform with respect to t of Eq. (4.21) must be taken to give

$$-\omega^2 \int_0^1 y(\eta, x)\, \tilde{u}(x, \omega)dx + \tilde{u}(x, \omega) = \tilde{F}(x, \omega) \qquad (4.25)$$

where

$\tilde{u}(x, \omega)$ is the Fourier transform of $u(x, t)$ } $u(x, 0) = u_t(x, 0) = 0$

$\tilde{F}(x, \omega)$ is the Fourier transform of $F(x, t)$ }

The solution of Eq. (4.22) is thus reduced to solving a typical inhomogeneous Fredholm[49, 50] integral equation and taking an inverse Fourier transform if necessary. There is a well known body[42, 49, 50] of theory available for the solution of Fredholm integral equations. The similarity between this theory and the theory used to solve the discrete matrix case is easily seen. Two of the most useful methods for solving the inhomogeneous Fredholm integral equations are based on iterative techniques of successive approximations. One method gives the solution in the form of a series, called the Liouville-Neuman Series, which can be shown to be a uniformly convergent series under certain conditions. However, in many practical cases this series cannot be summed in closed form and therefore only approximations to the solution can be obtained.

Fredholm, himself, noting the similarity between matrix problems and linear integral equations, devised the second method of solution. It can be used for both symmetric and non symmetric integral equations of the type given by Eq. (4.24). Fredholm's method involves the determination of an analytic function of α, $D(\alpha)$, called Fredholm's Determinant. $D(\alpha)$ may be calculated from a convergent power series in α. The coefficients of the powers of α in this series are determined from multi-dimensional integrals with a kernel related to the kernel of the integral equation. Finally, the simple Fredholm minor $D(\eta, x, \alpha)$ must be determined from another power series in α (with coefficients which are functions of η and x). Schematically the process of solution goes as follows:

Given the equation

$$a \int_0^1 y(\eta, x)\, u_1(x)dx + u_1(\eta) = F_1(\eta) \tag{4.26}$$

Construct

$$D(a) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} D_n a^n \; ; \; D(\eta, x, a) = y(\eta, x) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} D_n(\eta, x) a^n \tag{4.27}$$

$$D_m = \int_0^1 D_{m-1}(x, x)dx \tag{4.28}$$

$$D_m(\eta, x) = D_m\, y(\eta, x) - m \int_0^1 y(\eta, s) D_{m-1}(s, x)ds \tag{4.29}$$

$$D_o = 1; \qquad D_o(\eta, x) = y(\eta, x) \tag{4.30}$$

Starting with Eq. (4.29) $D_1(\eta, x)$ may be obtained by using $D_o(\eta, x)$ as given by Eq. (4.30). Likewise Eq. (4.28) gives D_1. Proceeding in this way the two series in Eq. (4.27) may be evaluated. In many practical cases these series have only a few non zero terms.

The solution to Eq. (4.26) is given by

$$u_1(\eta) = F_1(\eta) + a \int_0^1 \frac{D(\eta, x, a)}{D(a)} F_1(x)dx \tag{4.31}$$

provided $D(a) \neq 0$. If $D(a) = 0$, then a is an eigenvalue of the homogeneous equation ($F_1(x) = 0$).

One distinct advantage of the Fredholm method is that it applies for all a unless $D(a) = 0$. If $D(a) \neq 0$, the only solution to the homogeneous problem is the trivial one $u_1(\eta) = 0$. If $D(a) = 0$ then for

Eq. (4.26) to have a solution $F_1(\eta)$ must be orthogonal to the eigenfunctions of the adjoint integral equation associated with the eigenvalue $\overline{a}$ (complex conjugate of a).

To solve the problem in a directly analogous fashion to the discrete case, i.e., by first determining the eigenfunctions of the integral operator and superimposing the solutions of the homogeneous and inhomogeneous problems is quite tedious. However, it is important for analytical purposes to note that the Hilbert-Schmidt theorem guarantees the completeness of the set of eigenfunctions as far as acceptable functions in the domain of definition of the integral operator is concerned. The eigenfunctions of real symmetric linear integral operators are continuous, those associated with different eigenvalues are orthogonal and all the eigenvalues are real. The number of eigenfunctions of a real symmetric linear integral operator may be finite or a denumerable infinity. The number is finite only in the case of separable or degenerate kernels, i.e., only if $y(x, \eta)$ may be expanded in a *finite series*

$$y(x, \eta) = \sum_{n=1}^{N} a_n(x)\, b_n(\eta)$$

where $a_n(x)$ and $b_n(\eta)$ are functions of x and η, respectively. It may be seen from the Green's function that such kernels do not arise in passive systems.

From the point of view of the applied mathematician or the engineer, the existence of a complete theory of Fredholm integral equations is not sufficient. In the last section of this chapter some of

the available numerical approaches to integral equations will be reviewed.

The Integral Formulation of Linear Damped Continuous Systems

In line with the above work on the undamped system, the equations of motion of linear damped continuous systems may be written as

$$\int_0^1 m(x,\eta)\, u_{tt}(\eta,t)d + \int_0^1 c(x,\eta)u_t(\eta,t)d\eta + u(x,t) = f(x,t) \quad (4.32)$$

where $m(x,\eta)$ and $c(x,\eta)$ are the inertia and viscous damping influence functions, respectively. $m(x,\eta)$ may be interpreted as the displacement at point x due to a delta function of acceleration (normalized inertia force) at point η. In a similar manner, $c(x,\eta)$ is the displacement at point x due to a delta function of velocity (normalized viscous damping force) at point η. The actual determination of influence functions is quite difficult. In the fields of aeroelasticity[51] and structural analysis some work has been done on determining experimentally and analytically influence lines for specific problems. Influence lines are merely the form of the displacement of the entire structure due to a unit static delta function of force at a particular point. So far the author has not been able to show how to transform all damped self adjoint differential systems into symmetric integral systemsof type (4.32).

In the integral formulation of the continuous system, the boundary conditions are introduced into the problem through the in-

fluence functions. For passive systems the influence functions are real and symmetric. As in the differential formulation the concept of classical and non-classical systems is useful in separating passive systems into those systems capable of solution by direct expansion in terms of the eigenfunctions of the undamped problem and those requiring solution by Foss's method.

Necessary and Sufficient Conditions for Integral Systems to Be Classically Damped

The canonical form of the integral formulation of continuous systems may be written as

$$\int_0^1 m(x,\eta)u_{tt}(\eta,t)d\eta + \int_0^1 c(x,\eta)\, u_t(\eta,t)d\eta + u(x,t) = f(x,t) \qquad (4.33)$$

where $m(x,\eta)$ and $c(x,\eta)$ are symmetric functions in x and η.

It will now be shown that Eq. (4.33) possesses classical normal modes if and only if

$$\int_0^1 m(x,\eta)\, c(\eta,\xi)d\eta = \int_0^1 c(x,\eta)\, m(\eta,\xi)d\eta \qquad (4.34)$$

To show the sufficiency of the condition, consider the integral operator

$$\int_0^1 m(x,\eta)\, r(\eta)d\eta = g(x)$$

As $m(x,\eta)$ is a symmetric function in x and η, there exists a complete set of eigenfunctions $\phi_n(x)$, n=1, 2, . . . , such that

$$\int_0^1 \int_0^1 \phi_m(x)\, m(x, \eta)\, \phi_n(\eta)\, d\eta\, dx = 0 \qquad n \neq m$$
$$= 1 \qquad n = m \qquad (4.35)$$

On substituting

$$u(x, t) = \phi_n(x)\, e^{a_n t}$$

into the homogeneous undamped problem associated with Eq. (4.33)

$$\therefore \quad a_n^2 \int_0^1 m(x, \eta)\, \phi_n(\eta)\, d\eta = -\phi_n(x) \qquad (4.36)$$

Consider

$$\int_0^1 \int_0^1 \phi_m(x)\, c(x, \eta)\, \phi_n(\eta)\, dx\, d\eta = I_{nm} \qquad (4.37)$$

Substituting Eq. (4.36) into Eq. (4.37)

$$-a_n^2 \int_0^1 \int_0^1 \int_0^1 \phi_m(x)\, c(x, \eta)\, m(\eta, s)\, \phi_n(s)\, ds\, dx\, d\eta = I_{nm} \qquad (4.38)$$

On using the condition given by Eq. (4.34), Eq. (4.38) may be rewritten as

$$-a_n^2 \int_0^1 \int_0^1 \int_0^1 \phi_m(x)\, m(x, \eta)\, c(\eta, s)\, \phi_n(s)\, ds\, dx\, d\eta = I_{nm} \qquad (4.39)$$

Noting that

$$-\int_0^1 \phi_m(x)\, m(x, \eta)\, dx = \frac{1}{a_m^2}\, \phi_m(\eta)$$

Eq. (4.39) may be simplified to

$$\left[\frac{a_n^2}{a_m^2}\right]\int_0^1\int_0^1 \phi_m(\eta)\, c(\eta, s)\, \phi_n(s)\,ds\,d\eta = I_{nm} \qquad (4.40)$$

From Eqs. (4.39) and (4.40)

$$\left[1+\frac{a_n^2}{a_m^2}\right]\int_0^1\int_0^1 \phi_m(\eta)\, c(\eta, s)\, \phi_n(s)\,ds\,d\eta = 0$$

$$\therefore \text{ if } a_n^2 \neq a_m^2 \qquad m \neq n$$

$$\int_0^1\int_0^1 \phi_m(\eta)\, c(\eta, s)\, \phi_n(s)\,ds\,d\eta = 0$$

i. e., the system is classically damped.

If $a_n^2 = a_m^2$, there are some repeated eigenvalues associated with the undamped system. Suppose, for example there is an eigenvalue of multiplicity a with the following set of orthonormalized eigenfunctions

$$\phi_1(x), \quad \phi_2(x), \quad \ldots. \quad \phi_a(x) \qquad (4.41)$$

The system will be classical provided the set of orthonormalized eigenfunctions $\phi_1^*(x)$, $\phi_2^*(x)\ldots$ $\phi_a^*(x)$, obtained from linear combinations of the set given by Eq. (4.41), satisfy the following conditions

$$\int_0^1\int_0^1 \phi_m^*(\eta)\, c(\eta, x)\, \phi_r^*(x)\,dx\,d\eta = 0 \qquad n \neq m$$

where

$$\phi_m^*(x) = \sum_{j=1}^{a} a_{mj}\,\phi_j(x) \tag{4.42}$$

a_{mj} is the mj^{th} element of the matrix $[A]$. From the orthonormalization of the $\phi_m^*(x)$'s it may be seen that

$$A^T = A^{-1}$$

Consider

$$\int_0^1\int_0^1 \phi_m^*(\eta)\, c(\eta, x)\, \phi_n^*(x)\,dx\,d\eta = I_{nm}^* = \sum_{j=1}^{a}\sum_{\ell=1}^{a} a_{mj} I_{j\ell} a_{n\ell} \tag{4.43}$$

where $I_{j\ell}$ is defined by Eq. (4.37). But $\sum_{j=1}^{a}\sum_{j=1}^{a} a_{mj} I_{j\ell} a_{n\ell}$ is the mn^{th} term of $[A][I][A]^T$ where the ij^{th} element of $[I]$ is $I_{ij} = I_{ji}$. As $[I]$ is a symmetric matrix, $[A]$ exists such that

$$[A][I][A]^T = [\bar{I}] \quad , \text{ a diagonal matrix}$$

Thus by a suitable choice of $[A]$, the requirement specified by Eq. (4.42) for the system to be classical may be satisfied. Hence, the sufficiency of Eq. (4.34) for systems to be classical has been demonstrated.

The necessity of Eq. (4.34) for systems to be classical follows from the following considerations. If the system is classical,

$$\int_0^1\int_0^1 \phi_m(\eta)\, m(\eta, x)\, \phi_n(x)\,d\eta\; dx = 0 \qquad n \neq m$$

$$\int_0^1\int_0^1 \phi_m(\eta)\, c(\eta, x)\, \phi_n(x)\,d\eta\, dx = 0 \qquad n \neq m \tag{4.44}$$

$$\int_0^1 \phi_m(\eta)\,\phi_n(\eta)\,d\eta = 0 \qquad n \neq m$$
$$= 1 \qquad n = m$$

From Eq. (4.44) as the $\phi_m(x)$'s are complete and orthonormal

$$\int_0^1 c(x,\eta)\,\phi_n(\eta)\,d\eta = a_n \phi_n(x) \tag{4.45}$$

$$\int_0^1 m(x,\eta)\,\phi_n(\eta)\,d\eta = b_n \phi_n(x)$$

$$\therefore \int_0^1 \left\{ \int_0^1 c(x,\eta)\,\phi_n(\eta)\,d\eta \left\{ \int_0^1 m(x,\xi)\,\phi_n(\xi)\,d\xi \right\} \right\} dx$$

$$= \int_0^1 a_n b_n \phi_n^{\,2}(x)\,dx = a_n b_n \tag{4.46}$$

Interchanging dummy variables in (4.46)

$$\int_0^1 \left\{ \int_0^1 c(x,\xi)\,\phi_n(\xi)\,d\xi \left\{ \int_0^1 m(x,\eta)\,\phi_n(\eta)\,d\eta \right\} \right\} dx = a_n b_n \tag{4.47}$$

On subtracting Eq. (4.47) from Eq. (4.46)

$$\int_0^1 \int_0^1 \left\{ \phi_n(\eta)\,\phi_n(\xi) \left\{ \int_0^1 \left[c(x,\eta)\,m(x,\xi) - c(x,\xi)\,m(x,\eta) \right] dx \right\} \right\} d\eta\,d\xi = 0 \tag{4.48}$$

As Eq. (4.48) holds for all n

$$\therefore \quad \int_0^1 c(\eta, x)\, m(x, \xi)\, dx = \int_0^1 m(\eta, x)\, c(x, \xi)\, dx$$

Thus the condition given by Eq. (4. 34) is both a necessary and sufficient condition for systems with symmetric integral operators, to be classical. The analogy between the necessary and sufficient conditions for systems to be classical in the three different formulations of physical problems, namely discrete, differential and integral, is indeed striking.

Foss's Method Used with Non Classical Integral Systems

The equations of motion of continuous systems, in the integral formulation, may be written as

$$\int_0^1 m(x, \eta)\, u_{tt}(\eta, t)\, d\eta + \int_0^1 c(x, \eta)\, u_t(\eta, t)\, d\eta + u(x, t) = f(x, t) \qquad (4.49)$$

To Eq. (4. 49) add the identity

$$u_t(x, t) = z(x, t) \qquad (4.50)$$

Rewriting the homogeneous part of Eq. (4. 49)

$$\int_0^1 m(x, \eta)\, z_t(\eta, t)\, d\eta + \int_0^1 c(x, \eta)\, u_t(\eta, t)\, d + u(x, t) = 0 \qquad (4.51)$$

Let

$$u(x, t) = \phi_n(x)\, e^{\alpha_n t} \qquad (4.52)$$

$$z(x, t) = \beta_n(x)\, e^{\alpha_n t}$$

Substituting Eq. (4. 52) into Eqs. (4. 51) and (4. 50)

$$a_n \int_0^1 m(x, \eta)\beta_n(\eta)d\eta + a_n \int_0^1 c(x, \eta)\phi_n(\eta)d\eta + \phi_n(x) = 0 \qquad (4.53)$$

$$a_n \phi_n(x) = \beta_n(x) \qquad (4.54)$$

Equation (4. 53) is an eigenvalue-eigenfunction equation, giving the following set of eigenvalues and eigenfunctions

$$a_1, a_2 \ldots\ldots a_n \ldots\ldots$$

$$\phi_1(x), \phi_2(x) \ldots\ldots \phi_n(x) \ldots\ldots$$

$$\beta_1(x), \beta_2(x) \ldots\ldots \beta_n(x) \ldots\ldots$$

Multiply both sides of Eq. (4. 53) by $\phi_m(x)$ and both sides of Eq. (4. 54) by $\beta_m(x)$, add the resulting equations and integrate with respect to x between 0 and 1.

$$a_n \int_0^1 \int_0^1 \phi_m(x)\, m(x,\eta)\beta_n(\eta)d\eta\, dx + a_n \int_0^1 \int_0^1 \phi_m(x)\, c(x,\eta)\, \phi_n(\eta)\, d\eta\, dx$$

$$+ \int_0^1 \left\{\phi_n(x)\, \phi_m(x) + a_n \phi_n(x)\beta_m(x)\right\} dx = \int_0^1 \beta_n(x)\beta_m(x)dx \qquad (4.55)$$

Interchange the indices n and m in Eq. (4. 55)

$$a_m \int_0^1 \int_0^1 \phi_n(x) m(x,\eta)\beta_m(\eta)d\eta\, dx + a_m \int_0^1 \phi_n(x) c(x,\eta)\, \phi_m(\eta)\, d\eta\, dx$$

$$+ \int_0^1 \left\{\phi_n(x)\, \phi_m(x) + a_m \phi_m(x)\beta_n(x)\right\} dx = \int_0^1 \beta_n(x)\beta_m(x)dx \qquad (4.56)$$

On subtracting Eq. (4. 56) from Eq. (4. 55)

$$(a_n - a_m)\left\{\int_0^1\int_0^1 m(x,\eta)\left\{\phi_m(x)\beta_n(\eta) + \phi_n(x)\beta_m(\eta)\right\}dxd\eta + \int_0^1\int_0^1 \phi_n(x)\, c(x,\eta)\, \phi_m(\eta)dxd\eta\right\} = 0 \tag{4. 57}$$

$\therefore$ if $a_n \neq a_m$

$$\int_0^1\int_0^1 m(x,\eta)\left\{\phi_m(x)\beta_n(\eta) + \phi_n(x)\beta_m(\eta)\right\}dxd\eta + \int_0^1\int_0^1\left\{\phi_n(x)c(x,\eta)\,\phi_m(\eta)\right\}$$

$$dxd\eta = 0 \tag{4. 58}$$

On substituting Eq. (4. 58) into Eq. (4. 55)

$$-a_n\int_0^1\int_0^1 m(x,\eta)\,\phi_n(x)\beta_m(\eta)dxd\eta + (1+a_n a_m)\int_0^1 \phi_n(x)\,\phi_m(x)dx$$

$$= a_n a_m \int_0^1 \phi_n(x)\,\phi_m(x)dx \tag{4. 59}$$

or

$$\int_0^1\int_0^1 m(x,\eta)\beta_n(x)\beta_m(\eta)dxd\eta = \int_0^1 \phi_n(x)\,\phi_m(x)dx \tag{4. 60}$$

Equations (4. 60) and (4. 58) are the orthogonality relationships for the integral formulation of the continuous system.

The solution of the problem in the integral formulation by Foss's Method goes through in an analogous fashion to the solution in the differential formulation. The details will not be considered here.

Sufficient Conditions for Systems to be Classical

If $c(x, \eta)$ can be expanded in a series as follows

$$c(x,\eta) = a_o + a_1 m(x,\eta) + a_2 \int_0^1 m(x,\xi) m(\xi,\eta) d\xi + a_3 \int_0^1 \int_0^1 m(x,\xi) m(\xi, s)$$

$$m(s,\eta) d\xi\, ds + \ldots\ldots + a_n \underbrace{\int_0^1 \ldots\ldots \int_0^1}_{(n-1)\text{ fold}} \underbrace{m(x,\xi) m(\;) \ldots m(\;,\eta)}_{n\text{ factors}} d\xi\, d \ldots\ldots \qquad (4.61)$$

where the a_i, $i=1, 2, \ldots$, are constants, the system is classically damped. For, consider

$$\int_0^1 \phi_n(x) c(x,\eta)\, \phi_m(\eta) dx d\eta$$

$$= a_o \delta_n^m + a_1 \lambda_m^2 \delta_n^m + a_2 \lambda_m^4 \delta_n^m + \ldots\ldots + a_n \lambda_m^{2n} \delta_n^m \ldots\ldots$$

where $\delta_n^m = 0 \quad n \neq m$

$= 1 \quad n = m$

and the $\phi_n(x)$'s, $n=1, 2, \ldots$, are eigenfunctions of the undamped problem. Hence,

$$\int_0^1 \phi_n(x)\, c(x,\eta)\, \phi_m(\eta) dx d\eta = 0 \qquad n \neq m$$

$$= \sum_i a_i \lambda_m^{2i} \qquad n = m \qquad (4.63)$$

which shows that the system with a damping influence function given by Eq. (4.61) is classical. The series for $c(x, \eta)$ given by Eq. (4.61) is analogous to the Caughey series in the discrete case.

Numerical Methods in Continuous Systems

It was noted above that even in the case of classical systems, the actual determination of the eigenfunctions and eigenvalues of the system is in general quite a difficult problem. The number of problems which are exactly solvable, analytically, are very few, e.g., uniform beams, some uniform plate problems. The simplest possible generalization of the string problem (one dimensional wave equation), for example, a string with non uniform mass distribution, leads to an eigenvalue-eigenfunction equation which may or may not be solvable in closed form by analytical methods. Although it is very important to have a general theory of continuous systems, it must be realized that in practical applications the main emphasis is on the numerical analysis of the system.

Numerical analysis[52, 53] broadly defined involves the concept of approximate analysis. The closeness of the actual solution of the approximate problem to the exact solution of the original system constitutes the criterion of effectiveness of the numerical analysis. There are three main concepts in approximate analysis:

(i) Simplification of the model of the continuous structure, i.e., replacing the continuum by a multi-degree of freedom discrete structure which is then solved exactly.

(ii) Using the methods of the numerical analyst in an approximate solution of the differential or integral formulation of the problem, e.g., the finite differences method, the iteration methods, etc.

(iii) The application of variational principles, in which use is made of the minimum properties of certain functionals.

As (i) and (ii) are more suited to the type of problems of interest in this work, a brief discussion will first be given on (iii). The variational formulation of mechanics gives great insight into the underlying structure of the equations and provides the basis for the use of such techniques as the Rayleigh-Ritz approximation, the Rayleigh Quotient, Energy Methods, etc. The Rayleigh-Ritz approximation can be used to advantage in continuous systems. The method involves the selection of a set of functions which satisfy the boundary conditions of the problem. The approximate solution is then assumed to be a series of these functions with arbitrary coefficients. The coefficients are then selected to minimize or maximize a given integral and in this sense are best possible for the selected set of functions. The Rayleigh Quotient, which was also used in the discrete case, gives bounds for the eigenvalues of linear self adjoint operators. Any function satisfying the boundary conditions of the operator may be used in the Rayleigh Quotient. Naturally, the closer the function is to an eigenfunction the better the approximation to the eigenvalue. Quite often, the static deflection curve is used to determine the lowest eigenfrequency. Although many of these techniques were developed for the undamped problems, their extension to the approximate solution of the damped problem follows in an obvious manner.

Lumped Parameterization of Continuous Systems

Replacing the continuum by a discrete N degree of freedom system is a technique first used by Rayleigh.[4] It is based on physical rather than mathematical modeling in that the lumping of the parameters is done with the physical model in mind. The Myklestad-Prohl technique is an example of a very successful application of this approach to the vibration of beams. One question which always arises with any lumped parameterization scheme is what effect the degree of lumping (i.e., the number of degrees of freedom allowed) has on the accuracy of the solution. By replacing a continuous system by a discrete system, one is replacing functions by vectors and an infinite set of eigenvalues by a finite set. The question of accuracy is then twofold, namely, how close the N dimensional vector specifies the displacement of the N corresponding points of the continuous structure, and in what sense does a finite set of eigenvalues approximate an infinite set.

To illustrate some of the difficulties in the lumped parameterization of continuous systems the solution to an exactly solvable problem will be compared to the solution of its discrete approximation. The uniform elastic string in transverse vibration, originally solved by Lagrange with an excellent account of the analysis in Rayleigh's work,[4] provides an interesting example of this type of comparison. By considering a uniform elastic string of density ρ held by a tensile force T between two fixed points, a distance L apart, he shows that the eigenvalues and eigenfunctions of the continuous system are

$$\omega_r^{\infty} = \frac{r\pi}{L} \sqrt{\frac{T}{\rho}} \tag{4.64}$$

$$u_r^{\infty} = A_r^{\infty} \sin \frac{r\pi}{L} x \tag{4.65}$$

By considering the string to be composed of N sections, each of length L/N and placing one half the mass of the section at each end of the section, he derived the eigenvectors and eigenvalues of the resulting (N-1) degree of freedom system

$$\left\{U_r^N\right\} = A_r^N \begin{Bmatrix} \sin \frac{r\pi}{N} \\ \sin \frac{2r\pi}{N} \\ \vdots \\ \sin \frac{(N-1)r\pi}{N} \end{Bmatrix}; \quad \omega_r^N = \left\{\left(\frac{2N}{L}\right) \sin \frac{r\pi}{2N}\right\} \sqrt{\frac{T}{\rho}} \tag{4.66}$$

Hence, with this scheme of lumped parameterization, although the frequencies are in error, the normal modes for the discrete case are in fact correct, i.e., they give the exact displacement at the mass points. A measure of the error in the frequencies is

$$\frac{\omega_r^{\infty} - \omega_r^N}{\omega_r^{\infty}} = \frac{r\pi - 2N \sin \frac{r\pi}{2N}}{2N \sin \frac{r\pi}{2N}} \doteq \left(\frac{\pi^2}{24}\right)\left(\frac{r}{N}\right)^2 \quad \text{if } \left(\frac{r}{N}\right) << 1 \tag{4.67}$$

Duncan[55] uses a different scheme of lumped parameterization in that he replaces the mass of each segment by a concentrated mass at the centre. For the case of the uniform string Duncan's results are identical to those given above using Rayleigh's method of concentrating the masses. Two other problems in vibration lead to the same equation as the one dimensional wave equation considered here.

These are the torsional oscillation of a uniform elastic shaft and the longitudinal oscillation of a uniform and straight thin elastic rod. When using either Rayleigh's or Duncan's scheme of concentrating the mass (or moment of inertia in the case of torsional vibrations) the elastic reaction to the displacement of the concentrated mass must be the same as the elastic reaction of the continuous system at the same point. If these ideas are extended to the non uniform case it is easy to see that the Rayleigh method of mass concentration has some advantage when dealing with shafts formed from many uniform sections.

Frequency Error Law in Non-Uniform Vibration Problems

In the last section it was shown that the error in the natural frequencies of uniform continuous systems obeying the one dimensional wave equation, when calculated from the Rayleigh or Duncan discrete approximation, has the form

$$\varepsilon_{rN} = \frac{\omega_r^{\infty} - \omega_r^{N}}{\omega_r^{\infty}} \doteq \left(\frac{\pi^2}{24}\right)\left(\frac{r}{N}\right)^2 \quad \text{if} \quad \frac{r}{N} \ll 1 \tag{4.68}$$

i.e., it is $O\left(\left(\frac{r}{N}\right)^2\right)$ for (r/N) small. Now it will be shown that for non-uniform vibrating systems the error is still $O(1/N^2)$ for $(1/N)$ small.

Assume for the moment that the mode shapes of the discrete system correspond exactly at the mass concentration points to the eigenfunctions of the continuous system. This assumption will later be examined. The approach will be to use Rayleigh's Principle to determine the eigenvalues of the discrete system and to compare these

eigenvalues with the exact eigenvalues of the continuous system. The method will be applied here to the longitudinal vibrations of a non-uniform elastic rod but it may be used on any non-uniform system.

For convenience of notation, let the section lengths be 2h. The lumped parameterization method follows Duncan's approach of mass concentration in the center of each section. The kinetic energy of the structure, considered as a N degree of freedom system, vibrating in a pure mode may be written as

$$T_s = \frac{1}{2}\omega^2 \sum_{i=1}^{N} u^2(\xi_i) \int_{\xi_i-h}^{\xi_i+h} m(x)dx \tag{4.69}$$

$$\xi_i = (2i-1)h \quad ; \; 2hN = L$$

$$i=1, 2, \ldots N$$

The kinetic energy of the section of the continuous system, $\xi_i - h \leq x \leq \xi_i + h$ may be written as

$$\Delta T_{ic} = \frac{1}{2}\omega^2 \int_{\xi_i-h}^{\xi_i+h} m(x)\, u^2(x)dx \tag{4.70}$$

Expanding Eq. (4.70) by Taylor series

$$\Delta T_{ic} = \frac{1}{2}\omega^2 \int_{\xi_i-h}^{\xi_i+h} \Big[m(\xi_i)u^2(\xi_i) + (x-\xi_i)\{m'(\xi_i)u^2(\xi_i) + 2m(\xi_i)\, u(\xi_i)\, u'(\xi_i)\} + \frac{(x-\xi_i)^2}{2!}\{m''(\xi_i)u^2(\xi_i) + 4m'(\xi_i)u(\xi_i)u'(\xi_i) + 2m(\xi_i)\left|u'(\xi_i)\right|^2 + 2m(\xi_i)u(\xi_i)u''(\xi_i)\} + \ldots . \Big] dx$$

$$= \frac{1}{2}\omega^2\left\{2hm(\xi_i)u^2(\xi_i) + \frac{h^3}{3}\left\{m''(\xi_i)u^2(\xi_i) + 4m'(\xi_i)u(\xi_i)u'(\xi_i)\right.\right.$$

$$\left.\left. + 2m(\xi_i)\left|u'(\xi_i)\right|^2 + 2m(\xi_i)u(\xi_i)u''(\xi_i)\right\} + \ldots.\right\} \qquad (4.71)$$

The concentrated mass at ξ_i is

$$\int_{\xi_i-h}^{\xi_i+h} m(x)dx = \int_{\xi_i-h}^{\xi_i+h} \left\{m(\xi_i) + (x-\xi_i)m'(\xi_i) + \frac{(x-\xi_i)^2}{2!}m''(\xi_i) + \ldots.\right\} dx$$

$$= 2h\, m(\xi_i) + \frac{h^3}{3}\, m''(\xi_i) + \ldots. \qquad (4.72)$$

Hence

$$\Delta T_{ic} - \Delta T_{is} = \frac{\omega^2 h^3}{6}\left\{4m'(\xi_i)u(\xi_i)u'(\xi_i) + 2m(\xi_i)\left|u'(\xi_i)\right|^2\right.$$

$$\left. + 2m(\xi_i)u(\xi_i)u''(\xi_i)\right\} + O(h^5)$$

Therefore for the N sections

$$\frac{T_c - T_s}{\omega^2} = O(h^2) \qquad N = \frac{L}{2h} \qquad (4.73)$$

To calculate the potential energy (strain energy) let k(x) be the elastic stiffness at point x in the rod.

The stiffness element between $x=\xi_i$ and $x=\xi_{i+1}$, $k(\xi_i, \xi_{i+1})$ may be calculated from the following consideration, based on two springs in series

$$\frac{1}{k(\xi_i)} + \frac{1}{k(\xi_i, \xi_{i+1})} = \frac{1}{k(\xi_{i+1})} \qquad (4.74)$$

$$\therefore\ k(\xi_i, \xi_{i+1}) = \frac{k(\xi_{i+1})\, k(\xi_i)}{k(\xi_i) - k(\xi_{i+1})} \qquad (4.75)$$

The potential energy ΔV_{ic} of the corresponding section of the continuous system consists of two parts. The first part which is identical to Eq. (4.75), is the contribution to the potential energy of that part of the displacement which is derivable solely from consideration of statics (i.e., given that there is a relative displacement between the points $x=\xi_i$ and $x=\xi_{i+1}$ of $\left|u(\xi_{i+1})-u(\xi_i)\right|$ calculate the potential energy of the section from the consideration of a non-uniform massless spring). The second part of the potential energy of the section consists of the contribution of the inertia force due to the distributed mass. As the inertia load is proportional to h and the stiffness of the section is also proportional to h (ends assumed fixed), the relative displacement due to the inertia loads is proportional to h^2 and the addition to the potential energy is of order h^3. From these considerations,

$$\Delta V_{ic} - \Delta V_{is} = O(h^3)$$

$$\therefore \quad V_c - V_s = O(h^2)$$

From Rayleigh's Principle

$$\frac{V_s}{T_s} = \frac{V_c + O(h^2)}{T_c + O(h^2)} = \omega_o + O(h^2) = \omega_o + O\left(\frac{1}{N^2}\right) \qquad (4.76)$$

In order to complete the proof it is necessary to justify the initial assumption that the displacements at the mass concentration points are the same for the continuous system and its discrete approximation. This may be done by using the Lagrangian method of solution as follows.

Let $u_r(x)$, $r=1, 2, \ldots N$, be N displacement functions satisfying

the required boundary conditions of the problem. The mass influence functions m^c_{rs} may be calculated from

$$m^c_{rs} = \int_0^L m(x)\, u_r(x)\, u_s(x)dx \tag{4.77}$$

If a set of vectors defined as follows

$$\{U_i\} = \begin{Bmatrix} u_i(\xi_1) \\ u_i(\xi_2) \\ u_i(\xi_3) \\ \vdots \\ u_i(\xi_N) \end{Bmatrix} \quad i=1, 2, \ldots N \tag{4.78}$$

are used to determine the mass influence coefficients m^s_{rs} of the discrete system it is easy to see, using the expansion procedure developed above, that

$$m^c_{rs} - m^s_{rs} = O\left(\frac{1}{N^2}\right) \tag{4.79}$$

The stiffness influence coefficients are the same in the continuous and discrete systems as the elastic specification of the actual body is identical to that of the segmented body. Hence from Eq. (4.79) the difference between displacement of the actual body at the points $x = \xi_i$, $i=1, 2, \ldots N$, and the displacement of the corresponding mass concentration points in the discrete approximation is $O(1/N^2)$. But a difference in displacement of $O(1/N^2)$ can only effect the square of the frequencies obtained from the Rayleigh Quotient by $O(1/N^2)$. Hence the square of the natural frequencies of the discrete approximation are within $O(1/N^2)$

of the square of the actual frequencies of the continuous system.

In practical computations it is possible to use the foregoing rule to improve the accuracy of the estimates of the natural frequencies of the systems. Suppose ω is calculated from a discrete approximation of M and N sections (M, N large) then

$$\begin{aligned} \omega^2_N &= \omega^2_o + kN^{-2} \\ \omega^2_M &= \omega^2_o + kM^{-2} \\ \omega^2_o &= \frac{\omega^2_N M^{-2} - \omega^2_M N^{-2}}{M^{-2} - N^{-2}} \end{aligned} \tag{4.80}$$

If the mass concentration points were not in the middle of the section then the error in $T_c - T_s$ would be O(1/N). Consequently if the total mass of the section was concentrated at other than the center the estimate of the square of the natural frequencies would be only correct to O(1/N). However, by concentrating half the mass at each end of the section the error in the square of the frequency may be shown to be still of $O(1/N^2)$. In the literature a rough rule of 13 segments per complete wave length is generally considered as adequate for engineering applications.

It should be noted that, even if there is close agreement between the mode shapes and frequencies of the discrete and continuous system, there is no need for a similar correspondence in the stress distribution. Questions of stress and ultimate strength cannot be handled by the discretization process described above.

Gradwell[56] in a recent paper showed that for uniform beams

with clamped pinned or sliding ends the Rayleigh and the Duncan model lead to practically equal errors and the errors are of $O(1/N^4)$. For beams with one or both ends free the errors are of $O(1/N^2)$, and error (Duncan) = -1/2 error (Rayleigh).

Livesley[57] working on some beam vibration problems, noted a similar result with the Rayleigh model of a uniform beam simply supported. He concludes that the surprisingly small error is due to the fact that the modes for the discrete case are identical with those of the continuous system while at the same time the boundary conditions may be satisfied exactly. This small error could hardly be achieved with non uniform systems under general boundary conditions.

Solution of Continuous Systems by Numerical Methods

In contrast to the physical motivation for the lumped parameterization methods of the previous section, the numerical analysts approach the solution of continuous systems from a purely mathematical point of view. In the past significant work has been done by the numerical analysts[21, 24] on the special problems of mathematical physics. In the realm of partial differential equations, for example, various difference schemes have been used for the numerical solution of diffusion and heat flow problems, transport problems, wave propagation and elastic vibration problems, etc. Questions of stability and convergence of the solution of the resulting set of difference equations have been settled for certain classes of problems. However, for the general type of linear differential operators used in Chapter 3

very little work has been done on the effect of mesh size on the stability and convergence of the solutions of the difference equations.

Much work has been done on determining the eigenvalues and eigenfunctions of linear differential operators. Here again the question of error bounds has not been satisfactorily investigated. In practical engineering analysis two of the more usual methods of solution are the iteration method and the Galerkin method. The basis of all iteration methods is the selection of an initial solution which is in some sense close to the exact solution and the continued refinement of this solution by the iteration process until the resulting error is judged to be small. The Galerkin method expands the solution in a linear combination of known functions. The best linear combination is determined from a set of linear simultaneous equations.

In the last few years, great interest has developed in the numerical solution of integral equations. As in the case of partial differential equations the main emphasis has been on difference methods of solutions. Linear integral equations with symmetric kernels, like self adjoint linear differential equations, lead to symmetric difference equations and so in principle, may be solved by the methods of Chapters I and II. The question of error bounds on the solution of the difference equations approximating the integral equation (the error bound being the modulus of the maximum difference between the actual solution of the integral equation and the exact solution of the approximate problem) is an interesting one and has not been fully investigated. Wielandt,[54] in a recent paper has proposed a method for determining the eigen-

values and eigenfunctions of symmetric integral operators to within prespecified bounds. In a similar vein, Kantorovich and Krylov[58] have presented results on error bounds of the complete solution of symmetric integral equations.

In conclusion, it may be said that in the last decade a considerable amount of analytical work has been done on the numerical solution of the equations of mathematical physics. Methods that have been used in the past in a more or less heuristic manner by engineers and physicists have been subjected to critical analysis. It is to be hoped that in the near future, the questions on stability, convergence and error bounds will be settled.

SUMMARY AND CONCLUSIONS

The classical normal mode analysis, as applied to the vibration of damped lumped parameter multi-degree of freedom linear systems, was reviewed. It was shown by Caughey that the classical normal mode theory can be used to determine the response of systems, the damping matrices of which may be expanded in a particular power series of the matrices of the system. Systems with damping of this type are called classical systems and include as special cases those with Rayleigh type damping. Non-classical systems, if solvable by normal mode methods, must first be transformed to 2N-space and then the solution obtained using the results of Foss. However, not all non-classical systems can be so treated, and thus the justification of the general theory of vibration of linear dynamic systems as presented in this work.

To avoid unnecessary recourse to the general theory, a constructive necessary and sufficient condition for systems to be classical was presented. To simplify the analysis of weakly coupled non-classical systems some results from perturbation theory were derived. The stability of these systems was investigated using Liapunov's Direct Method. Whereas the normal mode analysis and its counterpart in the general theory – the generalized eigenvector -- gives valuable insight into the physical synthesis of the system it may not be the most efficient method for calculating the response of the system particularly when using high speed digital computers. Some discussion on this point

was presented and a useful algorithm for the determination of the response of these systems was given. A matter of considerable practical interest in this area is the determination of bounds for the natural frequency and coefficient of damping for each of the uncoupled modes of the system. Various results from matrix theory on the bounds of eigenvalues were presented and applied to the problem under discussion.

The second half of the thesis follows naturally from the first and is concerned with the response of continuous linear vibrating systems. In the continuous system one may use either an integral or a differential formulation. The concepts of classical and non-classical systems are still valid when dealing with continuous systems. Constructive necessary and sufficient conditions for continuous systems to be classical were derived. Whereas in the discrete systems exact solutions are always obtainable, although considerable computational effort may be required when the order of the system is high, this is not so, at the present time, in continuous systems. The reason for this is that not all linear differential or integral equations are solvable in closed form.

The numerical solution of continuous systems involve a discretization process. In effect the continuous system is represented as a lumped parameter system with many degrees of freedom. The literature is particularly poor on the approximations involved in the discretization of linear systems. Some comparisons between exactly solvable continuous systems and reasonable lumped parameter

approximations were given. A result of Duncan's for uniform parameters was noted and a discussion of the non-uniform case followed. The error bounds on the eigenvalues and the numerical solution of integral equations with symmetric kernals were discussed.

As a result of previous work by other investigators and this work, it may fairly be said that the general theory of lumped parameter linear damped systems is now fully developed. Whereas the theory of continuous linear systems is satisfactory for most practical purposes, some further results from numerical and functional analysis must be obtained before the theory will be fully developed.

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